FoMP: Vectors, Tensors and Fields

Contents

1 Vectors

1.1 Review of Vectors

1.1.1 Physics Terminology

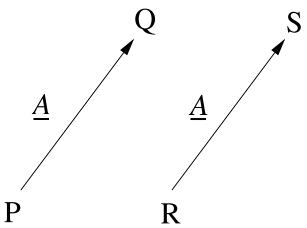
Scalar : quantity specified by a single number;

Vector : quantity specified by a number (magnitude) and a **direction**;

e.g. speed is a scalar, velocity is a vector

1.1.2 Geometrical Approach

A vector is *represented* by a 'directed line segment' with a length and direction proportional to the magnitude and direction of the vector (in appropriate units). A vector can be considered as a class of equivalent directed line segments *e.g.*

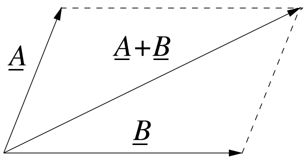


Both displacements from P to Q and from R to S are represented by the same vector. Also, different quantities can be represented by the same vector *e.g.* a displacement of A cm, or a velocity of A ms^{-1} or $\ldots$, where A is the magnitude or **length** of vector $\underline{A}$

Notation: Textbooks often denote vectors by boldface: $\boldsymbol{A}$ but here we use underline: $\underline{A}$ Denote a vector by $\underline{A}$ and its magnitude by $|\underline{A}|$ or A. *Always* underline a vector to distinguish it from its magnitude . A unit vector is often denoted by a hat $\underline{\hat{A}} = \underline{A} \,/\, A$ and represents a direction.

Addition of vectors—parallelogram law

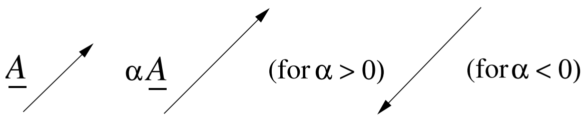
i.e.



$$\begin{aligned} \underline{A} + \underline{B} &= \underline{B} + \underline{A} \qquad \text{(commutative)} \; ; \\ (\underline{A} + \underline{B}) + \underline{C} &= \underline{A} + (\underline{B} + \underline{C}) \qquad \text{(associative)} \; . \end{aligned}$$

Multiplication by scalars,

A vector may be multiplied by a scalar to give a new vector *e.g.*



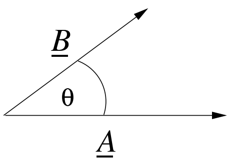
Also

$$
\begin{aligned}
|\alpha\underline{A}| &= |\alpha||\underline{A}| \\
\alpha(\underline{A}+\underline{B}) &= \alpha\underline{A}+\alpha\underline{B} \qquad \text{(distributive)} \\
\alpha(\beta\underline{A}) &= (\alpha\beta)\underline{A} \qquad \text{(associative)} \\
(\alpha+\beta)\underline{A} &= \alpha\underline{A}+\beta\underline{A}\,.
\end{aligned}
$$

1.1.3 Scalar or dot product

The scalar product (also known as the dot product) between two vectors is defined as

$$\boxed{(\underline{A}\cdot\underline{B}) \stackrel{\text{def}}{=} AB\cos\theta,\ \text{where } \theta \text{ is the angle between } \underline{A} \text{ and } \underline{B}}$$



$$\boxed{(\underline{A}\cdot\underline{B}) \text{ is a scalar — } \textit{i.e.} \text{ a single number.}}$$

Notes on scalar product

(*i*) $\underline{A}\cdot\underline{B} = \underline{B}\cdot\underline{A}$; $\underline{A}\cdot(\underline{B}+\underline{C}) = \underline{A}\cdot\underline{B}+\underline{A}\cdot\underline{C}$

(*ii*) $\underline{\hat{n}}\cdot\underline{A}$ = the scalar projection of $\underline{A}$ onto $\underline{\hat{n}}$, where $\underline{\hat{n}}$ is a unit vector

(*iii*) $(\underline{\hat{n}}\cdot\underline{A})\,\underline{\hat{n}}$ = the vector projection of $\underline{A}$ onto $\underline{\hat{n}}$

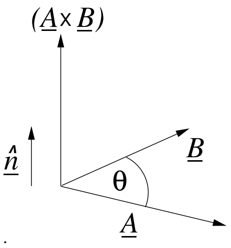
(*iv*) A vector may be resolved with respect to some direction $\underline{\hat{n}}$ into a parallel component $\underline{A}_{\parallel} = (\underline{\hat{n}}\cdot\underline{A})\underline{\hat{n}}$ and a perpendicular component $\underline{A}_{\perp} = \underline{A} - \underline{A}_{\parallel}$. You should check that $\underline{A}_{\perp}\cdot\underline{\hat{n}} = 0$

(*v*) $\boxed{\underline{A}\cdot\underline{A} = |\underline{A}|^2}$ which defines the magnitude of a vector. For a unit vector $\underline{\hat{A}}\cdot\underline{\hat{A}} = 1$

1.1.4 The vector or 'cross' product

$$\boxed{(\underline{A}\times\underline{B}) \stackrel{\text{def}}{=} AB\sin\theta\ \underline{\hat{n}}\ ,\ \text{where } \underline{\hat{n}} \text{ in the 'right-hand screw direction'}}$$

i.e. $\underline{\hat{n}}$ is a unit vector normal to the plane of $\underline{A}$ and $\underline{B}$, in the direction of a right-handed screw for rotation of $\underline{A}$ to $\underline{B}$ (through $< \pi$ radians).



$(\underline{A} \times \underline{B})$ is a vector — i.e. it has a direction and a length.

[It is also called the **cross** or **wedge** product — and in the latter case denoted by $\underline{A} \wedge \underline{B}$.]

Notes on vector product

(i) $\underline{A} \times \underline{B} = -\underline{B} \times \underline{A}$

(ii) $\underline{A} \times \underline{B} = 0$ if $\underline{A}, \underline{B}$ are parallel

(iii) $\underline{A} \times (\underline{B} + \underline{C}) = \underline{A} \times \underline{B} + \underline{A} \times \underline{C}$

(iv) $\underline{A} \times (\alpha \underline{B}) = \alpha \underline{A} \times \underline{B}$

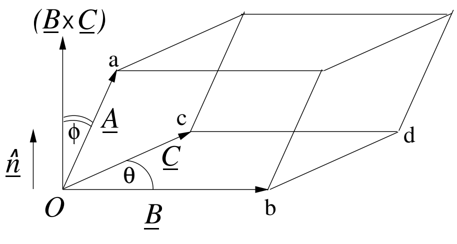
1.1.5 The Scalar Triple Product

The scalar triple product is defined as follows

$$(\underline{A}, \underline{B}, \underline{C}) \stackrel{\text{def}}{=} \underline{A} \cdot (\underline{B} \times \underline{C})$$

Notes

(i) If $\underline{A}$, $\underline{B}$ and $\underline{C}$ are three concurrent edges of a parallelepiped, the volume is $(\underline{A}, \underline{B}, \underline{C})$.
To see this, note that:



$$\begin{aligned}
\text{area of the base} &= \text{area of parallelogram } Obdc \\
&= B\,C\,\sin\theta = |\underline{B} \times \underline{C}| \\
\text{height} &= A\cos\phi = \underline{\hat{n}} \cdot \underline{A} \\
\text{volume} &= \text{area of base} \times \text{height} \\
&= B\,C\,\sin\theta\,\underline{\hat{n}} \cdot \underline{A} \\
&= \underline{A} \cdot (\underline{B} \times \underline{C})
\end{aligned}$$

(ii) If we choose $\underline{C}, \underline{A}$ to define the base then a similar calculation gives volume $= \underline{B} \cdot (\underline{C} \times \underline{A})$
We deduce the following symmetry/antisymmetry properties:

$$(\underline{A}, \underline{B}, \underline{C}) = (\underline{B}, \underline{C}, \underline{A}) = (\underline{C}, \underline{A}, \underline{B}) = -(\underline{A}, \underline{C}, \underline{B}) = -(\underline{B}, \underline{A}, \underline{C}) = -(\underline{C}, \underline{B}, \underline{A})$$

(iii) If $\underline{A}, \underline{B}$ and $\underline{C}$ are **coplanar** ($i.e.$ all three vectors lie in the same plane) then $V = (\underline{A}, \underline{B}, \underline{C}) = 0$, and vice-versa.

1.1.6 The Vector Triple Product

There are *several* ways of combining 3 vectors to form a new vector.
e.g. $\underline{A} \times (\underline{B} \times \underline{C})$; $(\underline{A} \times \underline{B}) \times \underline{C}$, etc. Note carefully that *brackets are important*, since

$$\underline{A} \times (\underline{B} \times \underline{C}) \;\neq\; (\underline{A} \times \underline{B}) \times \underline{C}\ .$$

Expressions involving two (or more) vector products can be simplified by using the identity:–

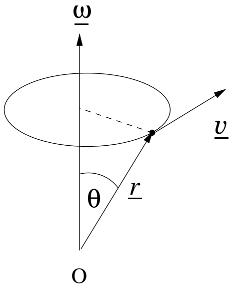
$$\boxed{\underline{A} \times (\underline{B} \times \underline{C}) \;=\; \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})}\ .$$

This is a result you must memorise. We will prove it later in the course.

1.1.7 Some examples in Physics

(i) Angular velocity

Consider a point in a rigid body rotating with **angular velocity** $\underline{\omega}$: $|\underline{\omega}|$ is the angular speed of rotation measured in radians per second and $\underline{\hat{\omega}}$ lies along the axis of rotation. Let the position vector of the point with respect to an origin O on the axis of rotation be $\underline{r}$.



You should convince yourself that $\underline{v} = \underline{\omega} \times \underline{r}$ by checking that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r \sin\theta = \omega \times$ radius of circle in which the point is travelling

(ii) Angular momentum

Now consider the **angular momentum** of the particle defined by $\underline{L} = \underline{r} \times (m\underline{v})$ where m is the mass of the particle.

Using the above expression for $\underline{v}$ we obtain

$$\underline{L} = m\underline{r} \times (\underline{\omega} \times \underline{r}) = m\left[\underline{\omega} r^2 - \underline{r}(\underline{r} \cdot \underline{\omega})\right]$$

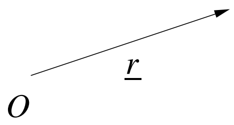
where we have used the identity for the vector triple product. Note that only if $\underline{r}$ is perpendicular to $\underline{\omega}$ do we obtain $\underline{L} = m\underline{\omega} r^2$, which means that only then are $\underline{L}$ and $\underline{\omega}$ in the same direction. Also note that $\underline{L} = 0$ if $\underline{\omega}$ and $\underline{r}$ are parallel.

end of lecture 1

1.2 Equations of Points, Lines and Planes

1.2.1 Position vectors

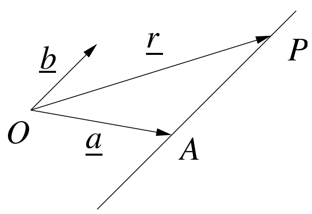
A **position vector** is a vector bound to some origin and gives the position of a point relative to that origin. It is often denoted $\underline{x}$ or $\underline{r}$.



The equation for a point is simply $\underline{r} = \underline{a}$ where $\underline{a}$ is some vector.

1.2.2 The Equation of a Line

Suppose that P lies on a line which passes through a point A which has a position vector $\underline{a}$ with respect to an origin O. Let P have position vector $\underline{r}$ relative to O and let $\underline{b}$ be a vector through the origin in a direction parallel to the line.



We may write

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

which is the **parametric equation of the line** *i.e.* as we vary the parameter λ from $-\infty$ to ∞, $\underline{r}$ describes all points on the line.

Rearranging and using $\underline{b} \times \underline{b} = 0$, we can also write this as:–

$$(\underline{r} - \underline{a}) \times \underline{b} \ = \ 0$$

or

$$\boxed{\underline{r} \times \underline{b} = \underline{c}}$$

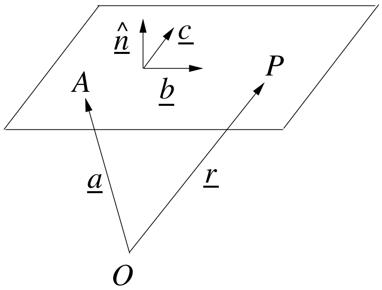
where $\underline{c} = \underline{a} \times \underline{b}$ is normal to the plane containing the line and origin.

Notes

(i) $\underline{r} \times \underline{b} = \underline{c}$ is an **implicit equation** for a line

(ii) $\underline{r} \times \underline{b} = 0$ is the equation of a line through the origin.

1.2.3 The Equation of a Plane



$\underline{r}$ is the position vector of an arbitrary point P on the plane
$\underline{a}$ is the position vector of a fixed point A in the plane
$\underline{b}$ and $\underline{c}$ are parallel to the plane but non-collinear: $\underline{b} \times \underline{c} \neq 0$.

We can express the vector $\underline{AP}$ in terms of $\underline{b}$ and $\underline{c}$, so that:

$$\underline{r} \;=\; \underline{a} + \underline{AP} \;=\; \underline{a} + \lambda \underline{b} + \mu \underline{c}$$

for some λ and μ. This is the **parametric equation of the plane**.

We define the unit normal to the plane

$$\hat{\underline{n}} \;=\; \frac{\underline{b} \times \underline{c}}{|\underline{b} \times \underline{c}|} \;.$$

Since $\underline{b} \cdot \hat{\underline{n}} = \underline{c} \cdot \hat{\underline{n}} = 0$, we have the implicit equation:–

$$(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} \;=\; 0 \;.$$

Alternatively, we can write this as:–

$$\boxed{\underline{r} \cdot \hat{\underline{n}} \;=\; p} \;,$$

where $p = \underline{a} \cdot \hat{\underline{n}}$ is the perpendicular distance of the plane from the origin.

This is a very important equation which you must be able to recognise.

Note: $\underline{r} \cdot \underline{a} \;=\; 0$ is the equation for a plane through the origin (with unit normal $\underline{a}/|\underline{a}|$).

1.2.4 Examples of Dealing with Vector Equations

Before going through some worked examples let us state two simple rules which will help you to avoid many common mistakes

1. **Always** check that the quantities on both sides of an equation are of the same type. e.g. any equation of the form *vector* = *scalar* is clearly wrong. (The only exception to this is if we lazily write *vector* = 0 when we mean $\underline{0}$.)

2. **Never** try to divide by a vector – there is no such operation!

Example 1: Is the following set of equations consistent?

$$\underline{r} \times \underline{b} = \underline{c} \tag{1}$$
$$\underline{r} = \underline{a} \times \underline{c} \tag{2}$$

Geometrical interpretation – the first equation is the (implicit) equation for a line whereas the second equation is the (explicit) equation for a point. Thus the question is whether the point is on the line. If we insert (2) into the l.h.s. of (1) we find

$$\underline{r} \times \underline{b} = (\underline{a} \times \underline{c}) \times \underline{b} = -\underline{b} \times (\underline{a} \times \underline{c}) = -\underline{a}\,(\underline{b} \cdot \underline{c}) + \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{3}$$

Now from (1) we have that $\underline{b} \cdot \underline{c} = \underline{b} \cdot (\underline{r} \times \underline{b}) = 0$ thus (3) becomes

$$\underline{r} \times \underline{b} = \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{4}$$

so that, on comparing (1) and (4), we require

$$\underline{a} \cdot \underline{b} = 1$$

for the equations to be consistent.

Example 2: Solve the following set of equations for $\underline{r}$.

$$\underline{r} \times \underline{a} = \underline{b} \tag{5}$$
$$\underline{r} \times \underline{c} = \underline{d} \tag{6}$$

Geometrical interpretation – both equations are equations for lines *e.g.* (5) is for a line parallel to $\underline{a}$ where $\underline{b}$ is normal to the plane containing the line and the origin. The problem is to find the intersection of two lines. (Here we assume the equations are consistent and the lines do indeed have an intersection).

Consider

$$\underline{b} \times \underline{d} = (\underline{r} \times \underline{a}) \times \underline{d} = -\underline{d} \times (\underline{r} \times \underline{a}) = -\underline{r}\,(\underline{a} \cdot \underline{d}) + \underline{a}\,(\underline{d} \cdot \underline{r})$$

which is obtained by taking the vector product of l.h.s of (5) with $\underline{d}$.

Now from (6) we see that $\underline{d} \cdot \underline{r} = \underline{r} \cdot (\underline{r} \times \underline{c}) = 0$. Thus

$$\underline{r} = -\frac{\underline{b} \times \underline{d}}{\underline{a} \cdot \underline{d}} \quad \text{for} \quad \underline{a} \cdot \underline{d} \neq 0 \,.$$

Alternatively we could have taken the vector product of the l.h.s. of (6) with $\underline{b}$ to find

$$\underline{b} \times \underline{d} = \underline{b} \times (\underline{r} \times \underline{c}) = \underline{r}\,(\underline{b} \cdot \underline{c}) - \underline{c}\,(\underline{b} \cdot \underline{r}) \,.$$

Since $\underline{b} \cdot \underline{r} = 0$ we find

$$\underline{r} = \frac{\underline{b} \times \underline{d}}{\underline{b} \cdot \underline{c}} \quad \text{for} \quad \underline{b} \cdot \underline{c} \neq 0 \, .$$

It can be checked from (5) and (6) and the properties of the scalar triple product that for the equations to be consistent $\underline{b} \cdot \underline{c} = -\underline{d} \cdot \underline{a}$. Hence the two expressions derived for $\underline{r}$ are the same.

What happens when $\underline{a} \cdot \underline{d} = \underline{b} \cdot \underline{c} = 0$? In this case the above approach does not give an expression for $\underline{r}$. However from (6) we see $\underline{a} \cdot \underline{d} = 0$ implies that $\underline{a} \cdot (\underline{r} \times \underline{c}) = 0$ so that $\underline{a}$, $\underline{c}$, $\underline{r}$ are coplanar. We can therefore write $\underline{r}$ as a linear combination of $\underline{a}, \underline{c}$:

$$\underline{r} = \alpha \, \underline{a} + \gamma \, \underline{c} \, . \tag{7}$$

To determine the scalar α we can take the vector product with $\underline{c}$ to find

$$\underline{d} = \alpha \, \underline{a} \times \underline{c} \tag{8}$$

(since $\underline{r} \times \underline{c} = \underline{d}$ from (6) and $\underline{c} \times \underline{c} = 0$). In order to extract α we need to convert the vectors in (8) into scalars. We do this by taking, for example, a scalar product with $\underline{b}$

$$\underline{b} \cdot \underline{d} = \alpha \, \underline{b} \cdot (\underline{a} \times \underline{c})$$

so that

$$\alpha = -\frac{\underline{b} \cdot \underline{d}}{(\underline{a} \, , \, \underline{b} \, , \, \underline{c})} \, .$$

Similarly, one can determine γ by taking the vector product of (7) with $\underline{a}$:

$$\underline{b} = \gamma \, \underline{c} \times \underline{a}$$

then taking a scalar product with $\underline{b}$ to obtain finally

$$\gamma = \frac{\underline{b} \cdot \underline{b}}{(\underline{a} \, , \, \underline{b} \, , \, \underline{c})} \, .$$

Example 3: Solve for $\underline{r}$ the vector equation

$$\underline{r} + (\underline{\hat{n}} \cdot \underline{r}) \, \underline{\hat{n}} + 2\underline{\hat{n}} \times \underline{r} + 2\underline{b} = 0 \tag{9}$$

where $\underline{\hat{n}} \cdot \underline{\hat{n}} = 1$.

In order to unravel this equation we can try taking scalar and vector products of the equation with the vectors involved. However straight away we see that taking various products with $\underline{r}$ will not help, since it will produce terms that are quadratic in $\underline{r}$. Instead, we want to eliminate $(\hat{\underline{n}} \cdot \underline{r})$ and $\hat{\underline{n}} \times \underline{r}$ so we try taking scalar and vector products with $\hat{\underline{n}}$. Taking the scalar product one finds

$$\hat{\underline{n}} \cdot \underline{r} + (\hat{\underline{n}} \cdot \underline{r})(\hat{\underline{n}} \cdot \hat{\underline{n}}) + 0 + 2\hat{\underline{n}} \cdot \underline{b} = 0$$

so that, since $(\hat{\underline{n}} \cdot \hat{\underline{n}}) = 1$, we have

$$\hat{\underline{n}} \cdot \underline{r} = -\hat{\underline{n}} \cdot \underline{b} \tag{10}$$

Taking the vector product of (9) with $\hat{\underline{n}}$ gives

$$\hat{\underline{n}} \times \underline{r} + 0 + 2\left[\, \hat{\underline{n}}(\hat{\underline{n}} \cdot \underline{r}) - \underline{r} \,\right] + 2\hat{\underline{n}} \times \underline{b} = 0$$

so that

$$\hat{\underline{n}} \times \underline{r} = 2\left[\, \hat{\underline{n}}(\underline{b} \cdot \hat{\underline{n}}) + \underline{r} \,\right] - 2\hat{\underline{n}} \times \underline{b} \tag{11}$$

where we have used (10). Substituting (10) and (11) into (9) one eventually obtains

$$\underline{r} = \frac{1}{5}\left[\, -3(\underline{b} \cdot \hat{\underline{n}})\,\hat{\underline{n}} + 4(\hat{\underline{n}} \times \underline{b}) - 2\underline{b} \,\right] \tag{12}$$

end of lecture 2

1.3 Vector Spaces and Orthonormal Bases

1.3.1 Review of vector spaces

Let V denote a vector space. Then vectors in V obey the following rules for addition and multiplication by scalars

$$\begin{aligned} \underline{A} + \underline{B} &\in V \qquad \text{if} \quad \underline{A}, \underline{B} \in V \\ \alpha \underline{A} &\in V \qquad \text{if} \quad \underline{A} \in V \\ \alpha(\underline{A} + \underline{B}) &= \alpha\underline{A} + \alpha\underline{B} \\ (\alpha + \beta)\underline{A} &= \alpha\underline{A} + \beta\underline{A} \end{aligned}$$

The space contains a zero vector or null vector, $\underline{0}$, so that, for example $(\underline{A}) + (-\underline{A}) = \underline{0}$.

Of course as we have seen, vectors in $\mathbb{R}^3$ (usual 3-dimensional real space) obey these axioms. Other simple examples are a plane through the origin which forms a two-dimensional space and a line through the origin which forms a one-dimensional space.

1.3.2 Linear Independence

Consider two vectors $\underline{A}$ and $\underline{B}$ in a plane through the origin and the equation:–

$$\boxed{\alpha\underline{A} + \beta\underline{B} = 0} \; .$$

If this is satisfied for *non-zero* α and β then $\underline{A}$ and $\underline{B}$ are said to be **linearly dependent**.

$$i.e.\;\; \underline{B} = -\frac{\alpha}{\beta}\underline{A} \; .$$

Clearly $\underline{A}$ and $\underline{B}$ are **collinear** (either parallel or anti-parallel). If this equation can be satisfied *only* for $\alpha = \beta = 0$, then $\underline{A}$ and $\underline{B}$ are **linearly independent**, and obviously *not collinear* (*i.e.* no λ can be found such that $\underline{B} = \lambda\underline{A}$).

Notes

(i) If $\underline{A}$, $\underline{B}$ are linearly independent any vector $\underline{r}$ in the plane may be written uniquely as a linear combination

$$\underline{r} = a\underline{A} + b\underline{B}$$

(ii) We say $\underline{A}$, $\underline{B}$ span the plane or $\underline{A}$, $\underline{B}$ form a basis for the plane

(iii) We call (a, b) a representation of $\underline{r}$ in the basis formed by $\underline{A}$, $\underline{B}$ and a, b are the components of $\underline{r}$ in this basis.

In 3 dimensions three vectors are linearly dependent if we can find non-trivial α, β, γ (i.e. not all zero) such that

$$\boxed{\alpha\underline{A} + \beta\underline{B} + \gamma\underline{C} = 0}$$

otherwise $\underline{A}, \underline{B}, \underline{C}$ are linearly independent (no one is a linear combination of the other two).

Notes

(i) If $\underline{A}$, $\underline{B}$ and $\underline{C}$ are linearly independent they span $\mathbb{R}^3$ and form a basis *i.e.* for any vector $\underline{r}$ we can find scalars a, b, c such that

$$\underline{r} \;=\; a\underline{A} + b\underline{B} + c\underline{C} \; .$$

(ii) The **triple** of numbers (a, b, c) is the **representation** of $\underline{r}$ in this basis; a, b, c are said to be the components of $\underline{r}$ in this basis.

(*iii*) The geometrical interpretation of linear dependence in three dimensions is that

$$\boxed{\text{three linearly dependent vectors} \Leftrightarrow \text{three coplanar vectors}}$$

To see this note that if $\alpha \underline{A} + \beta \underline{B} + \gamma \underline{C} = 0$ then

$$\begin{array}{ll} \alpha \neq 0 & \alpha \underline{A} \cdot (\underline{B} \times \underline{C}) = 0 \Rightarrow \underline{A}, \underline{B}, \underline{C} \quad \text{are coplanar} \\ \alpha = 0 & \text{then } \underline{B} \text{ is collinear with } \underline{C} \text{ and } \underline{A}, \underline{B}, \underline{C} \text{ are coplanar} \end{array}$$

These ideas can be generalised to vector spaces of arbitrary dimension. For a space of dimension n one can find at most n linearly independent vectors.

1.3.3 Standard orthonormal basis: Cartesian basis

A basis in which the basis vectors are *orthogonal* and *normalised* (of unit length) is called an **orthonormal** basis.

You have already have encountered the idea of *Cartesian coordinates* in which points in space are labelled by coordinates (x, y, z). We introduce orthonormal basis vectors denoted by either $\underline{i}$, $\underline{j}$ and $\underline{k}$ or $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ which point along the x, y and z-axes. It is usually understood that the basis vectors are related by the r.h. screw rule, with $\underline{i} \times \underline{j} = \underline{k}$ and so on, cyclically.

In the 'xyz' notation the components of a vector $\underline{A}$ are A_x, A_y, A_z, and a vector is written in terms of the basis vectors as

$$\underline{A} \;=\; A_x\, \underline{i} + A_y\, \underline{j} + A_z\, \underline{k} \quad \text{or} \quad \underline{A} = A_x\, \underline{e}_x + A_y\, \underline{e}_y + A_z\, \underline{e}_z\,.$$

Also note that in this basis, the basis vectors themselves are represented by

$$\underline{i} = \underline{e}_x = (1, 0, 0) \quad \underline{j} = \underline{e}_y = (0, 1, 0) \quad \underline{k} = \underline{e}_z = (0, 0, 1)$$

1.3.4 Suffix or Index notation

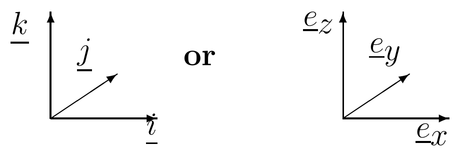
A more systematic labelling of orthonormal basis vectors for $I\!R^3$ is by $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. i.e. instead of $\underline{i}$ we write $\underline{e}_1$, instead of $\underline{j}$ we write $\underline{e}_2$, instead of $\underline{k}$ we write $\underline{e}_3$. Then

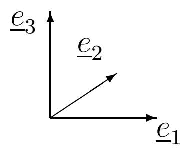
$$\boxed{\underline{e}_1 \cdot \underline{e}_1 = \underline{e}_2 \cdot \underline{e}_2 = \underline{e}_3 \cdot \underline{e}_3 = 1; \quad \underline{e}_1 \cdot \underline{e}_2 = \underline{e}_2 \cdot \underline{e}_3 = \underline{e}_3 \cdot \underline{e}_1 = 0.} \tag{13}$$

Similarly the components of any vector $\underline{A}$ in 3-d space are denoted by A_1, A_2 and A_3.

This scheme is known as the **suffix** notation. Its great advantages over 'xyz' notation are that it clearly generalises easily to any number of dimensions and greatly simplifies manipulations and the verification of various identities (see later in the course).

Old Notation **New Notation**





$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ $\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ $\underline{r} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$

Thus any vector $\underline{A}$ is written in this new notation as

$$\underline{A} \; = A_1 \; \underline{e}_1 + A_2 \; \underline{e}_2 + A_3 \; \underline{e}_3 = \sum_{i=1}^{3} \; A_i \; \underline{e}_i \; .$$

The final summation will often be abbreviated to $\underline{A} = \sum_i A_i \; \underline{e}_i$.

Notes

(i) The numbers A_i are called the **(Cartesian) components** (or representation) of $\underline{A}$ with respect to the basis set $\{\underline{e}_i\}$.

(ii) We may write $\underline{A} = \sum_{i=1}^{3} A_i \, \underline{e}_i = \sum_{j=1}^{3} A_j \, \underline{e}_j = \sum_{\alpha=1}^{3} A_\alpha \, \underline{e}_\alpha$ where i, j and α are known as summation or 'dummy' indices.

(iii) The components are obtained by using the orthonormality properties of equation (13):

$$\underline{A} \cdot \underline{e}_1 \;\; = \;\; (A_1\underline{e}_1 + A_2\underline{e}_2 + A_3\underline{e}_3) \cdot \underline{e}_1 = A_1$$

A_1 is the projection of $\underline{A}$ in the direction of $\underline{e}_1$.

Similarly for the components A_2 and A_3. So in general we may write

$\underline{A} \cdot \underline{e}_i = A_i$ or sometimes $(\underline{A})_i$

where in this equation i is a 'free' index and may take values $i = 1, 2, 3$. In this way we are in fact condensing three equations into one.

(iv) In terms of these components, the scalar product takes on the form:–

$$\boxed{\underline{A} \cdot \underline{B} = \sum_{i=1}^{3} A_i \, B_i \; .}$$

end of lecture 3

1.4 Suffix Notation

1.4.1 Free Indices and Summation Indices

Consider, for example, the vector equation

$$\underline{a} - (\underline{b} \cdot \underline{c})\, \underline{d} + 3\underline{n} = 0 \tag{14}$$

As the basis vectors are linearly independent the equation must hold for each component:

$$a_i - (\underline{b} \cdot \underline{c})\, d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3 \tag{15}$$

The free index i occurs **once and only once** in each term of the equation. In general every term in the equation must be of the same kind *i.e.* have the same free indices.

Now suppose that we want to write the scalar product that appears in the second term of equation (15) in suffix notation. As we have seen summation indices are 'dummy' indices and can be relabelled

$$\underline{b} \cdot \underline{c} = \sum_{i=1}^{3} b_i c_i = \sum_{k=1}^{3} b_k c_k$$

This freedom should *always* be used to avoid confusion with other indices in the equation. Thus we avoid using i as a summation index, as we have already used it as a free index, and write equation (15) as

$$a_i - \left(\sum_{k=1}^{3} b_k c_k \right) d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

rather than

$$a_i - \left(\sum_{i=1}^{3} b_i c_i \right) d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

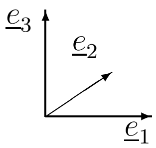
which would lead to great confusion, inevitably leading to mistakes, when the brackets are removed!

1.4.2 Handedness of Basis

In the usual Cartesian basis that we have considerd up to now, the basis vectors $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ form a *right-handed* basis, that is, $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$ and $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$.

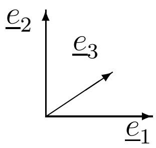
However, we could choose $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_3$, and so on, in which case the basis is said to be *left-handed*.

right handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_1 \times \underline{e}_2 \\ \underline{e}_1 &= \underline{e}_2 \times \underline{e}_3 \\ \underline{e}_2 &= \underline{e}_3 \times \underline{e}_1 \\ &\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = 1} \end{aligned}$$

left handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_2 \times \underline{e}_1 \\ \underline{e}_1 &= \underline{e}_3 \times \underline{e}_2 \\ \underline{e}_2 &= \underline{e}_1 \times \underline{e}_3 \\ &\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = -1} \end{aligned}$$

1.4.3 The Vector Product in a right-handed basis

$$\underline{A} \times \underline{B} = (\sum_{i=1}^{3} A_i\, \underline{e}_i) \times (\sum_{j=1}^{3} B_j\, \underline{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i\, B_j\, (\underline{e}_i \times \underline{e}_j)\, .$$

Since $\underline{e}_1 \times \underline{e}_1 = \underline{e}_2 \times \underline{e}_2 = \underline{e}_3 \times \underline{e}_3 = 0$, and $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_2 \times \underline{e}_1 = \underline{e}_3$, etc. we have

$$\underline{A} \times \underline{B} = \underline{e}_1(A_2B_3 - A_3B_2) + \underline{e}_2(A_3B_1 - A_1B_3) + \underline{e}_3(A_1B_2 - A_2B_1) \qquad (16)$$

from which we deduce that

$$(\underline{A} \times \underline{B})_1 \;=\; (A_2B_3 - A_3B_2)\ \text{, etc.}$$

Notice that the right-hand side of equation (16) corresponds to the expansion of the determinant

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

by the first row.

It is now easy to write down an expression for the scalar triple product

$$\begin{aligned}
\underline{A} \cdot (\underline{B} \times \underline{C}) &= \sum_{i=1}^{3} A_i (\underline{B} \times \underline{C})_i \\
&= A_1(B_2C_3 - C_2B_3) - A_2(B_1C_3 - C_1B_3) + A_3(B_1C_2 - C_1B_2) \\
&= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} .
\end{aligned}$$

The symmetry properties of the scalar triple product may be deduced from this by noting that interchanging two rows (or columns) changes the value by a factor -1.

1.4.4 Summary of algebraic approach to vectors

We are now able to define vectors and the various products of vectors in an algebraic way (as opposed to the geometrical approach of lectures 1 and 2).

A **vector** is *represented* (in some orthonormal basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$) by an ordered set of 3 numbers with certain laws of addition.

$$\begin{aligned}
\textit{e.g.}\ \underline{A} &\quad \text{is represented by} \quad (A_1, A_2, A_3)\,; \\
\underline{A} + \underline{B} &\quad \text{is represented by} \quad (A_1 + B_1, A_2 + B_2, A_3 + B_3)\,.
\end{aligned}$$

The various 'products' of vectors are defined as follows:–

The Scalar Product is denoted by $\underline{A} \cdot \underline{B}$ and **defined** as:–

$$\underline{A} \cdot \underline{B} \stackrel{\text{def}}{=} \sum_i A_i B_i \,.$$

$$\underline{A} \cdot \underline{A} = A^2 \text{ defines the magnitude } A \text{ of the vector.}$$

The Vector Product is denoted by $\underline{A} \times \underline{B}$, and is **defined** in a right-handed basis as:–

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} .$$

The Scalar Triple Product

$$\begin{aligned}
(\underline{A}, \underline{B}, \underline{C}) &\stackrel{\text{def}}{=} \sum_i A_i (\underline{B} \times \underline{C})_i \\
&= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} .
\end{aligned}$$

In all the above formula the summations imply sums over each index taking values $1, 2, 3$.

1.4.5 The Kronecker Delta Symbol δ_{ij}

We define the symbol δ_{ij} (pronounced "delta i j"), where i and j can take on the values 1 to 3, such that

$$\boxed{\begin{aligned} \delta_{ij} &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}}$$

i.e. $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{12} = \delta_{13} = \delta_{23} = \cdots = 0$.

The equations satisfied by the **orthonormal basis vectors** $\underline{e}_i$ can all now be written as:–

$$\boxed{\underline{e}_i \cdot \underline{e}_j = \delta_{ij}}\ .$$

e.g. $\underline{e}_1 \cdot \underline{e}_2 = \delta_{12} = 0 \quad ; \quad \underline{e}_1 \cdot \underline{e}_1 = \delta_{11} = 1$ **Notes**

(i) Since there are two free indices i and j, $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ is equivalent to 9 equations

(ii) $\delta_{ij} = \delta_{ji}$ [*i.e.* δ_{ij} is *symmetric* in its indices.]

(iii) $\displaystyle\sum_{i=1}^{3} \delta_{ii} = 3 \quad (= \delta_{11} + \delta_{22} + \delta_{33})$

(iv) $\displaystyle\sum_{j=1}^{3} A_j \delta_{jk} = A_1\delta_{1k} + A_2\delta_{2k} + A_3\delta_{3k}$

Remember that k is a free index. Thus if $k = 1$ then only the first term on the rhs contributes and rhs $= A_1$, similarly if $k = 2$ then rhs $= A_2$ and if $k = 2$ then rhs $= A_3$. Thus we conclude that

$$\boxed{\sum_{j=1}^{3} A_j \delta_{jk} = A_k}$$

In other words, the Kronecker delta picks out the kth term in the sum over j. This is in particular true for the multiplication of two Kronecker deltas:

$$\sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = \delta_{ik}$$

Generalising the reasoning in (iv) implies the so-called **sifting property**:

$$\boxed{\sum_{j=1}^{3} (\text{anything })_j \delta_{jk} = (\text{anything })_k}$$

where $(\text{anything})_j$ denotes any expression that has a single free index j.

Examples of the use of this symbol are:–

1.
$$\begin{aligned} \underline{A} \cdot \underline{e}_j &= (\sum_{i=1}^{3} A_i \underline{e}_i) \cdot \underline{e}_j = \sum_{i=1}^{3} A_i \, (\underline{e}_i \cdot \underline{e}_j) \\ &= \sum_{i=1}^{3} A_i \delta_{ij} = A_j, \quad \text{since terms with } i \neq j \text{ vanish.} \end{aligned}$$

2.
$$\begin{aligned} \underline{A} \cdot \underline{B} &= (\sum_{i=1}^{3} A_i \underline{e}_i) \cdot (\sum_{j=1}^{3} B_j \underline{e}_j) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \, (\underline{e}_i \cdot \underline{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij} \\ &= \sum_{i=1}^{3} A_i B_i \quad (\text{ or } \sum_{j=1}^{3} A_j B_j). \end{aligned}$$

1.4.6 Matrix representation of δ_{ij}

We may label the elements of a (3×3) matrix M as M_{ij},

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

Thus we see that if we write δ_{ij} as a matrix we find that it is the identity matrix $\mathbb{1}$.

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

end of lecture 4

1.5 More About Suffix Notation

1.5.1 Einstein Summation Convention

As you will have noticed, the novelty of writing out summations as in Lecture 4 soon wears thin. A way to avoid this tedium is to adopt the Einstein summation convention; by adhering strictly to the following rules the summation signs are suppressed.

Rules

(i) Omit summation signs

(ii) If a suffix appears twice, a summation is implied *e.g.* $A_iB_i = A_1B_1 + A_2B_2 + A_3B_3$
Here i is a *dummy* index.

(iii) If a suffix appears only once it can take any value *e.g.* $A_i = B_i$ holds for $i = 1, 2, 3$
Here i is a *free* index. Note that there may be more than one free index. **Always** check that the free indices match on both sides of an equation *e.g.* $A_j = B_i$ is WRONG.

(iv) A given suffix **must not** appear more than twice in any term of an expression. Again, **always** check that there are no multiple indices *e.g.* $A_iB_iC_i$ is WRONG.

Examples

$$\underline{A} = A_i\underline{e}_i \qquad \textit{here } i \textit{ is a dummy index.}$$

$$\underline{A}\cdot\underline{e}_j = A_i\underline{e}_i\cdot\underline{e}_j = A_i\delta_{ij} = A_j \qquad \textit{here } i \textit{ is a dummy index but } j \textit{ is a free index.}$$

$$\underline{A}\cdot\underline{B} = (A_i\underline{e}_i)\cdot(B_j\underline{e}_j) = A_iB_j\delta_{ij} = A_jB_j \qquad \textit{here i,j are dummy indices.}$$

$$(\underline{A}\cdot\underline{B})(\underline{A}\cdot\underline{C}) = A_iB_iA_jC_j \qquad \textit{again i,j are dummy indices.}$$

Armed with the summation convention one can rewrite many of the equations of the previous lecture without summation signs *e.g.* the sifting property of δ_{ij} now becomes

$$\boxed{[\cdots]_j\delta_{jk} \ = [\cdots]_k}$$

so that, for example, $\delta_{ij}\delta_{jk} = \delta_{ik}$

From now on, except where indicated, the summation convention will be assumed. You should make sure that you are completely at ease with it.

1.5.2 Levi-Civita Symbol ϵ_{ijk}

We saw in the last lecture how δ_{ij} could be used to greatly simplify the writing out of the orthonormality condition on basis vectors.

We seek to make a similar simplification for the vector products of basis vectors (taken here to be right handed) *i.e.* we seek a simple, uniform way of writing the equations

$$\underline{e}_1\times\underline{e}_2 = \underline{e}_3 \quad \underline{e}_2\times\underline{e}_3 = \underline{e}_1 \quad \underline{e}_3\times\underline{e}_1 = \underline{e}_2$$
$$\underline{e}_1\times\underline{e}_1 = 0 \quad \underline{e}_2\times\underline{e}_2 = 0 \quad \underline{e}_3\times\underline{e}_3 = 0$$

To do so we define the Levi-Cevita symbol ϵ_{ijk} (pronounced 'epsilon i j k'), where i, j and k can take on the values 1 to 3, such that:–

$$\boxed{\begin{aligned}\epsilon_{ijk} &= +1 \text{ if } ijk \text{ is an } \textit{even} \text{ permutation of } 123\;;\\ &= -1 \text{ if } ijk \text{ is an } \textit{odd} \text{ permutation of } 123\;;\\ &= 0 \text{ otherwise } (\textit{i.e.}\ 2 \text{ or more indices are the same})\;.\end{aligned}}$$

An *even* permutation consists of an *even* number of transpositions.
An *odd* permutations consists of an *odd* number of transpositions.

$$\begin{aligned}\text{For example,}\quad \epsilon_{123} &= +1\,;\\ \epsilon_{213} &= -1\ \{\text{ since } (123) \to (213) \text{ under } \textit{one} \text{ transposition } [1 \leftrightarrow 2]\}\,;\\ \epsilon_{312} &= +1\ \{(123) \to (132) \to (312);\ 2 \text{ transpositions; } [2 \leftrightarrow 3][1 \leftrightarrow 3]\}\,;\\ \epsilon_{113} &= 0\,;\quad \epsilon_{111} = 0\,;\ \text{etc.}\end{aligned}$$

$$\boxed{\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1\ ;\quad \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1\ ;\quad \text{all others} = 0\ .}$$

1.5.3 Vector product

The equations satisfied by the vector products of the (right-handed) orthonormal basis vectors $\underline{e}_i$ can now be written uniformly as :–

$$\boxed{\underline{e}_i \times \underline{e}_j \;=\; \epsilon_{ijk}\,\underline{e}_k} \qquad (i, j = 1,2,3)\ .$$

For example,

$$\underline{e}_1 \times \underline{e}_2 = \epsilon_{121}\,\underline{e}_1 + \epsilon_{122}\,\underline{e}_2 + \epsilon_{123}\,\underline{e}_3 = \underline{e}_3 \quad;\quad \underline{e}_1 \times \underline{e}_1 = \epsilon_{111}\,\underline{e}_1 + \epsilon_{112}\,\underline{e}_2 + \epsilon_{113}\,\underline{e}_3 = 0$$

Also,

$$\begin{aligned}\underline{A} \times \underline{B} &= A_i\,B_j\,\underline{e}_i \times \underline{e}_j\\ &= \epsilon_{ijk}\,A_i B_j\,\underline{e}_k\end{aligned}$$

but,

$$\underline{A} \times \underline{B} = (\underline{A} \times \underline{B})_k\,\underline{e}_k\ .$$

Thus

$$\boxed{(\underline{A} \times \underline{B})_k \;=\; \epsilon_{ijk}\,A_i B_j}$$

Always recall that we are using the summation convention. For example writing out the sums

$$\begin{aligned}(\underline{A} \times \underline{B})_3 &= \epsilon_{113}\, A_1 B_1 + \epsilon_{123}\, A_2 B_3 + \epsilon_{133}\, A_3 B_3 + \cdots \\ &= \epsilon_{123}\, A_1 B_2 + \epsilon_{213}\, A_2 B_1 \qquad \text{(only non-zero terms)} \\ &= A_1 B_2 - A_2 B_1\end{aligned}$$

Now note a 'cyclic symmetry' of ϵ_{ijk}

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

This holds for *any* choice of i, j and k. To understand this note that

1. If any pair of the free indices i, j, k are the same, all terms vanish;
2. If (ijk) is an even (odd) permutation of (123), then so is (jki) and (kij), but (jik), (ikj) and (kji) are odd (even) permutations of (123).

Now use the cyclic symmetry to find alternative forms for the components of the vector product

$$\begin{aligned}(\underline{A} \times \underline{B})_k &= \epsilon_{ijk}\, A_i B_j = \epsilon_{kij}\, A_i B_j \\ \text{or relabelling indices} \qquad & k \to i \quad i \to j \quad j \to k \\ (\underline{A} \times \underline{B})_i &= \epsilon_{jki}\, A_j B_k = \epsilon_{ijk}\, A_j B_k\ .\end{aligned}$$

The scalar triple product can also be written using ϵ_{ijk}

$$(\underline{A}, \underline{B}, \underline{C}) = \underline{A} \cdot (\underline{B} \times \underline{C}) = A_i (\underline{B} \times \underline{C})_i$$

$$\boxed{(\underline{A}, \underline{B}, \underline{C}) = \epsilon_{ijk}\, A_i B_j C_k\ .}$$

Now as an exercise in index manipulation we can prove the cyclic symmetry of the scalar product

$$\begin{aligned}(\underline{A}, \underline{B}, \underline{C}) &= \epsilon_{ijk}\, A_i B_j C_k \\ &= -\epsilon_{ikj}\, A_i B_j C_k \qquad \text{interchanging two indices of } \epsilon_{ikj} \\ &= +\epsilon_{kij}\, A_i B_j C_k \qquad \text{interchanging two indices again} \\ &= \epsilon_{ijk}\, A_j B_k C_i \qquad \text{relabelling indices} \quad k \to i \quad i \to j \quad j \to k \\ &= \epsilon_{ijk}\, C_i A_j B_k = (\underline{C}, \underline{A}, \underline{B})\end{aligned}$$

1.5.4 Product of two Levi-Civita symbols

We state without formal proof the following identity (see questions on Problem Sheet 3)

$$\boxed{\epsilon_{ijk}\,\epsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}.}$$

To verify this is true one can check all possible cases *e.g.* $\epsilon_{12k}\,\epsilon_{12k} = \epsilon_{121}\,\epsilon_{121} + \epsilon_{122}\,\epsilon_{122} + \epsilon_{123}\,\epsilon_{123} = 1 = \delta_{11}\delta_{22} - \delta_{12}\delta_{21}$. More generally, note that the left hand side of the boxed equation may be written out as

- $\epsilon_{ij1}\,\epsilon_{rs1} + \epsilon_{ij2}\,\epsilon_{rs2} + \epsilon_{ij3}\,\epsilon_{rs3}$ where i, j, r, s are free indices;
- for this to be non-zero we must have $i \neq j$ and $r \neq s$
- only one term of the three in the sum can be non-zero ;
- if $i = r$ and $j = s$ we have $+1$; if $i = s$ and $j = r$ we have -1 .

The product identity furnishes an algebraic proof for the 'BAC-CAB' rule. Consider the i^{th} component of $\underline{A} \times (\underline{B} \times \underline{C})$:

$$\begin{aligned}
\left[\underline{A} \times (\underline{B} \times \underline{C})\right]_i &= \epsilon_{ijk}\, A_j (\underline{B} \times \underline{C})_k \\
&= \epsilon_{ijk}\, A_j\, \epsilon_{krs}\, B_r C_s \;=\; \epsilon_{ijk}\, \epsilon_{rsk}\, A_j B_r C_s \\
&= (\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})\, A_j B_r C_s \\
&= (A_j B_i C_j - A_j B_j C_i) \\
&= B_i (\underline{A} \cdot \underline{C}) - C_i (\underline{A} \cdot \underline{B}) \\
&= \left[\underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})\right]_i
\end{aligned}$$

Since i is a free index we have proven the identity for all three components $i = 1, 2, 3$.

end of lecture 5

1.6 Change of Basis

1.6.1 Linear Transformation of Basis

Suppose $\{\underline{e}_i\}$ and $\{\underline{e}_i{}'\}$ are two different orthonormal bases. How do we relate them?

Clearly $\underline{e}_1{}'$ can be written as a linear combination of the vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$. Let us write the linear combination as

$$\underline{e}_1{}' = \lambda_{11}\underline{e}_1 + \lambda_{12}\underline{e}_2 + \lambda_{13}\underline{e}_3$$

with similar expressions for $\underline{e}_2{}'$ and $\underline{e}_3{}'$. In summary,

$$\boxed{\underline{e}_i{}' \ = \lambda_{ij}\, \underline{e}_j} \tag{17}$$

(assuming summation convention) where λ_{ij} $(i = 1, 2, 3$ and $j = 1, 2, 3)$ are the 9 numbers relating the basis vectors $\underline{e}_1{}'$, $\underline{e}_2{}'$ and $\underline{e}_3{}'$ to the basis vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$.

Notes

(i) Since $\underline{e}_i{}'$ are orthonormal

$$\underline{e}_i{}' \cdot \underline{e}_j{}' \ = \ \delta_{ij} \ .$$

Now the l.h.s. of this equation may be written as

$$(\lambda_{ik}\, \underline{e}_k) \cdot (\lambda_{jl}\, \underline{e}_l) = \lambda_{ik}\lambda_{jl}\, (\underline{e}_k \cdot \underline{e}_l) = \lambda_{ik}\lambda_{jl}\delta_{kl} = \lambda_{ik}\, \lambda_{jk}$$

(in the final step we have used the sifting property of δ_{kl}) and we deduce

$$\boxed{\lambda_{ik}\lambda_{jk} \ = \ \delta_{ij}} \tag{18}$$

(ii) In order to determine λ_{ij} from the two bases consider

$$\underline{e}_i{}' \cdot \underline{e}_j \ = \ (\lambda_{ik}\, \underline{e}_k) \cdot \underline{e}_j = \lambda_{ik}\, \delta_{kj} = \lambda_{ij} \ .$$

Thus

$$\boxed{\underline{e}_i{}' \cdot \underline{e}_j = \lambda_{ij}} \tag{19}$$

1.6.2 Inverse Relations

Consider expressing the unprimed basis in terms of the primed basis and suppose that

$$\underline{e}_i = \mu_{ij}\, \underline{e}_j{}'.$$

Then $\quad \lambda_{ki} \ = \ \underline{e}_k{}' \cdot \underline{e}_i \ = \mu_{ij}\, (\underline{e}_k{}' \cdot \underline{e}_j{}') \ = \mu_{ij}\, \delta_{kj} = \mu_{ik} \quad$ so that

$$\boxed{\mu_{ij} = \lambda_{ji}} \tag{20}$$

Note that $\underline{e}_i \cdot \underline{e}_j \ = \ \delta_{ij} \ = \lambda_{ki}\, (\underline{e}_k{}' \cdot \underline{e}_j) \ = \lambda_{ki}\lambda_{kj}$ and so we obtain a second relation

$$\boxed{\lambda_{ki}\lambda_{kj} \ = \ \delta_{ij}} \ . \tag{21}$$

1.6.3 The Transformation Matrix

We may label the elements of a 3×3 matrix M as M_{ij}, where i labels the row and j labels the column in which M_{ij} appears:

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

The summation convention can be used to describe matrix multiplication. The ij component of a product of two 3×3 matrices M, N is given by

$$(MN)_{ij} = M_{i1}\,N_{1j} + M_{i2}\,N_{2j} + M_{i3}\,N_{3j} = M_{ik}\,N_{kj} \tag{22}$$

Likewise, recalling the definition of the transpose of a matrix $(M^T)_{ij} = M_{ji}$

$$(M^T N)_{ij} = (M^T)_{ik} N_{kj} = M_{ki} N_{kj} \tag{23}$$

We can thus arrange the numbers λ_{ij} as elements of a square matrix, denoted by λ and known as the **transformation matrix**:

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

We denote the matrix transpose by λ^T and define it by $(\lambda^T)_{ij} = \lambda_{ji}$ so we see from equation (20) that $\mu = \lambda^T$ is the transformation matrix for the inverse transformation.

We also note that δ_{ij} may be thought of as elements of a 3×3 unit matrix:

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{33} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}.$$

i.e. the matrix representation of the Kronecker delta symbol is the **unit matrix** $\mathbb{1}$.

Comparing equation(18) with equation (22), and equation (21) with equation (23), we see that the relations $\lambda_{ik}\lambda_{jk} = \lambda_{ki}\lambda_{kj} = \delta_{ij}$ can be written in matrix notation as:-

$$\boxed{\lambda\lambda^T = \lambda^T\lambda = \mathbb{1}}\;, \quad i.e. \;\boxed{\lambda^{-1} = \lambda^T}\;.$$

This is the condition for an **orthogonal matrix** and the transformation (from the $\underline{e}_i$ basis to the $\underline{e}_i{}'$ basis) is called an **orthogonal transformation**.

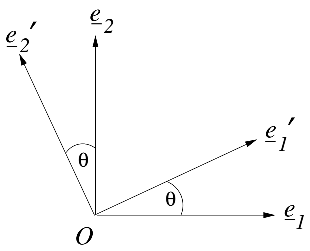
Now from the properties of determinants, $|\lambda\lambda^T| = |\mathbb{1}| = 1 = |\lambda|\,|\lambda^T|$ and $|\lambda^T| = |\lambda|$, we have that $|\lambda|^2 = 1$ hence

$$|\lambda| \;=\; \pm 1\ .$$

If $|\lambda| = +1$ the orthogonal transformation is said to be 'proper' If $|\lambda| = -1$ the orthogonal transformation is said to be 'improper'

1.6.4 Examples of Orthogonal Transformations

Rotation about the $\underline{e}_3$ axis. We have $\underline{e}_3{}' = \underline{e}_3$ and thus for a rotation through θ,



$$\begin{aligned}
\underline{e}_3{}'\cdot\underline{e}_1 &= \underline{e}_1{}'\cdot\underline{e}_3 = \underline{e}_3{}'\cdot\underline{e}_2 = \underline{e}_2{}'\cdot\underline{e}_3 = 0\,, \quad \underline{e}_3{}'\cdot\underline{e}_3 = 1\\
\underline{e}_1{}'\cdot\underline{e}_1 &= \cos\theta\\
\underline{e}_1{}'\cdot\underline{e}_2 &= \cos(\pi/2-\theta) \;=\; \sin\theta\\
\underline{e}_2{}'\cdot\underline{e}_2 &= \cos\theta\\
\underline{e}_2{}'\cdot\underline{e}_1 &= \cos(\pi/2+\theta) \;=\; -\sin\theta
\end{aligned}$$

Thus

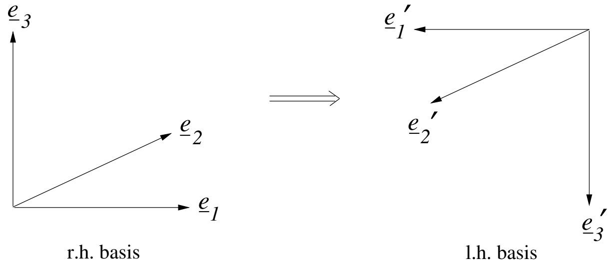
$$\lambda \;=\; \begin{pmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $\lambda\lambda^T = \mathbb{1}$. Since $|\lambda| = \cos^2\theta + \sin^2\theta = 1$, this is a *proper* transformation. Note that rotations cannot change the handedness of the basis vectors.

Inversion or Parity transformation. This is defined such that $\underline{e}_i{}' = -\underline{e}_i$.

$$\text{i.e. } \lambda_{ij} \;=\; -\delta_{ij} \qquad \text{or} \qquad \lambda \;=\; \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1 \end{pmatrix} \;=\; -\mathbb{1}\ .$$

Clearly $\lambda\lambda^T = \mathbb{1}$. Since $|\lambda| = -1$, this is an *improper* transformation. Note that the handedness of the basis is reversed: $\underline{e}_1{}' \times \underline{e}_2{}' = -\underline{e}_3{}'$



Reflection. Consider reflection of the axes in $\underline{e}_2$–$\underline{e}_3$ plane so that $\underline{e}_1{}' = -\underline{e}_1$, $\underline{e}_2{}' = \underline{e}_2$ and $\underline{e}_3{}' = \underline{e}_3$. The transformation matrix is:–

$$\lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $|\lambda| = -1$, this is an *improper* transformation. Again the handedness of the basis changes.

1.6.5 Products of Transformations

Consider a transformation λ to the basis $\{\underline{e}_i{}'\}$ followed by a transformation ρ to another basis $\{\underline{e}_i{}''\}$

$$\underline{e}_i \overset{\lambda}{\Longrightarrow} \underline{e}_i{}' \overset{\rho}{\Longrightarrow} \underline{e}_i{}''$$

Clearly there must be an orthogonal transformation $\underline{e}_i \overset{\xi}{\Longrightarrow} \underline{e}_i{}''$

Now

$$\underline{e}_i{}'' = \rho_{ij}\underline{e}_j{}' = \rho_{ij}\lambda_{jk}\underline{e}_k = (\rho\lambda)_{ik}\underline{e}_k \quad \text{so} \quad \boxed{\xi = \rho\lambda}$$

Notes

(*i*) Note the order of the product: the matrix corresponding to the first change of basis stands to the right of that for the second change of basis. In general, transformations do not commute so that $\rho\lambda \neq \lambda\rho$.

(*ii*) The inversion and the identity transformations commute with all transformations.

1.6.6 Improper Transformations

We may write any improper transformation ξ (for which $|\xi| = -1$) as $\xi = (-\mathbb{1})\,\lambda$ where $\lambda = -\xi$ and $|\lambda| = +1$ Thus an improper transformation can always be expressed as a proper transformation followed by an inversion.

e.g. consider ξ for a reflection in the $1-3$ plane which may be written as

$$\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Identifying λ from $\xi = (-\mathbb{1})\,\lambda$ we see that λ is a rotation of π about $\underline{e}_2$.

1.6.7 Summary

If $|\lambda| = +1$ we have a **proper** orthogonal transformation which is equivalent to rotation of axes. It can be proven that any rotation is a proper orthogonal transformation and vice-versa.

If $|\lambda| = -1$ we have an **improper** orthogonal transformation which is equivalent to rotation of axes then inversion. This is known as an improper rotation since it *changes the handedness of the basis.*

end of lecture 6

1.7 Transformation Properties of Vectors and Scalars

1.7.1 Transformation of vector components

Let $\underline{A}$ be any vector, with components A_i in the basis $\{\underline{e}_i\}$ and A'_i in the basis $\{\underline{e}_i{}'\}$ *i.e.*

$$\boxed{\underline{A} = A_i\, \underline{e}_i = A'_i\, \underline{e}_i{}'}\ .$$

The components are related as follows, taking care with dummy indices:–

$$A'_i = \underline{A} \cdot \underline{e}_i{}' = (A_j \underline{e}_j) \cdot \underline{e}_i{}' = (\underline{e}_i{}' \cdot \underline{e}_j) A_j = \lambda_{ij} A_j$$

$$\boxed{A'_i = \lambda_{ij} A_j}$$

$$A_i = \underline{A} \cdot \underline{e}_i = (A'_k \underline{e}_k{}') \cdot \underline{e}_i = \lambda_{ki} A'_k = (\lambda^T)_{ik} A'_k .$$

Note carefully that we do *not* put a prime on the vector itself – there is only one vector, $\underline{A}$, in the above discussion.

However, the *components* of this vector are different in different bases, and so are denoted by A_i in the basis $\{\underline{e}_i\}$, A'_i in the basis $\{\underline{e}_i{}'\}$, etc.

In matrix form we can write these relations as

$$\boxed{\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}}$$

Example: Consider a rotation of the axes about $\underline{e}_3$

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\, A_1 + \sin\theta\, A_2 \\ \cos\theta\, A_2 - \sin\theta\, A_1 \\ A_3 \end{pmatrix}$$

A direct check of this using trigonometric considerations is significantly harder!

1.7.2 The Transformation of the Scalar Product

Let $\underline{A}$ and $\underline{B}$ be vectors with components A_i and B_i in the basis $\{\underline{e}_i\}$ and components A_i' and B_i' in the basis $\{\underline{e}_i{}'\}$. In the basis $\{\underline{e}_i\}$, the scalar product, denoted by $(\underline{A} \cdot \underline{B})$, is:–

$$(\underline{A} \cdot \underline{B}) \;= A_i B_i \,.$$

In the basis $\{\underline{e}_i{}'\}$, we denote the scalar product by $(\underline{A} \cdot \underline{B})\,'$, and we have

$$\begin{aligned} (\underline{A} \cdot \underline{B})\,' &= A_i' B_i' \;= \lambda_{ij} A_j \lambda_{ik} B_k \;= \delta_{jk} A_j B_k \\ &= A_j B_j = (\underline{A} \cdot \underline{B})\,. \end{aligned}$$

Thus the scalar product is the same evaluated in any basis. This is of course expected from the geometrical definition of scalar product which is independent of basis. We say that the scalar product is *invariant* under a change of basis.

Summary We have now obtained an algebraic definition of scalar and vector quantities. Under the orthogonal transformation from the basis $\{\underline{e}_i\}$ to the basis $\{\underline{e}_i{}'\}$, defined by the transformation matrix λ : $\underline{e}_i{}' = \lambda_{ij}\, \underline{e}_j$, we have that:–

- A **scalar** is a single number ϕ which is invariant:

  $$\boxed{\phi' \;=\; \phi}\,.$$

  Of course, not all scalar quantities in physics are expressible as the scalar product of two vectors *e.g.* mass, temperature.

- A **vector** is an 'ordered triple' of numbers A_i which transforms to A_i' :

  $$\boxed{A_i' \;= \lambda_{ij} A_j}\,.$$

1.7.3 Summary of story so far

We take the opportunity to summarise some key-points of what we have done so far. N.B. this is NOT a list of everything you need to know.

Key points from geometrical approach

You should recognise on sight that

$$\begin{aligned} \underline{r} \times \underline{b} = \underline{c} &\qquad \text{is a line } (\underline{r} \text{ lies on a line}) \\ \underline{r} \cdot \underline{a} = d &\qquad \text{is a plane } (\underline{r} \text{ lies in a plane}) \end{aligned}$$

Useful properties of scalar and vector products to remember

$$\begin{aligned} \underline{a} \cdot \underline{b} = 0 &\quad \Leftrightarrow \quad \text{vectors orthogonal} \\ \underline{a} \times \underline{b} = 0 &\quad \Leftrightarrow \quad \text{vectors collinear} \\ \underline{a} \cdot (\underline{b} \times \underline{c}) = 0 &\quad \Leftrightarrow \quad \text{vectors co-planar or linearly dependent} \\ \underline{a} \times (\underline{b} \times \underline{c}) &\quad = \quad \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \end{aligned}$$

Key points of suffix notation

We label orthonormal basis vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ and write the expansion of a vector $\underline{A}$ as

$$\underline{A} = \sum_{i=1}^{3} A_i \underline{e}_i$$

The Kronecker delta δ_{ij} can be used to express the orthonormality of the basis

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

The Kronecker delta has a very useful sifting property

$$\sum_j [\cdots]_j \delta_{jk} = [\cdots]_k$$

$$(\underline{e}_1, \underline{e}_2, \underline{e}_3) = \pm 1 \quad \text{determines whether the basis is right- or left-handed}$$

Key points of summation convention

Using the summation convention we have for example

$$\underline{A} = A_i \underline{e}_i$$

and the sifting property of δ_{ij} becomes

$$[\cdots]_j \delta_{jk} = [\cdots]_k$$

We introduce ϵ_{ijk} to enable us to write the vector products of basis vectors in a r.h. basis in a uniform way

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k$$

.

The vector products and scalar triple products in a r.h. basis are

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad \text{or equivalently} \quad (\underline{A} \times \underline{B})_i \;=\; \epsilon_{ijk} A_j B_k$$

$$\underline{A} \cdot (\underline{B} \times \underline{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad \text{or equivalently} \quad \underline{A} \cdot (\underline{B} \times \underline{C}) \;=\; \epsilon_{ijk} A_i B_j C_k$$

Key points of change of basis

The new basis is written in terms of the old through

$$\underline{e}_i{}' = \lambda_{ij} \underline{e}_j \quad \text{where } \lambda_{ij} \text{ are elements of a } 3 \times 3 \text{ transformation matrix } \lambda$$

λ is an orthogonal matrix, the defining property of which is $\lambda^{-1} = \lambda^T$ and this can be written as

$$\lambda \lambda^T = \mathbb{1} \quad \text{or} \quad \lambda_{ik} \lambda_{jk} = \delta_{ij}$$

$|\lambda| = \pm 1$ decides whether the transformation is proper or improper i.e. whether the handedness of the basis is changed

Key points of algebraic approach

A **scalar** is defined as a number that is invariant under an orthogonal transformation

A **vector** is defined as an object $\underline{A}$ represented in a basis by numbers A_i which transform to A'_i through

$$A'_i \;= \lambda_{ij} A_j.$$

or in matrix form

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

end of lecture 7

2 Tensors

2.1 Tensors of Second Rank

2.1.1 Nature of Physical Laws

The simplest physical laws are expressed in terms of scalar quantities which are independent of our choice of basis *e.g.* the gas law

$$pV = RT$$

relating pressure, volume and temperature.

At the next level of complexity are laws relating vector quantities:

$$\begin{aligned} \underline{F} &= m\underline{a} && \text{Newton's Law} \\ \underline{J} &= g\underline{E} && \text{Ohm's Law, } g \text{ is conductivity} \end{aligned}$$

Notes

(i) These laws take the form *vector* = *scalar* × *vector*

(ii) They relate two vectors in the *same* direction

If we consider Newton's Law, for instance, then in a particular Cartesian basis $\{\underline{e}_i\}$, $\underline{a}$ is represented by its components $\{a_i\}$ and $\underline{F}$ by its components $\{F_i\}$ and we can write

$$F_i = m\, a_i$$

In another such basis $\{\underline{e}_i{}'\}$

$$F'_i = m\, a'_i$$

where the set of numbers, $\{a'_i\}$, is in general different from the set $\{a_i\}$. Likewise, the set $\{F'_i\}$ differs from the set $\{F_i\}$, but of course

$$a'_i = \lambda_{ij} a_j \quad \text{and} \quad F'_i = \lambda_{ij} F_j$$

Thus we can think of $\underline{F} = m\underline{a}$ as representing an infinite set of relations between measured components in various bases. Because all vectors transform the same way under orthogonal transformations, the relations have the *same form* in all bases. We say that Newton's Law, expressed in component form, is *form invariant* or *covariant.*

2.1.2 Examples of more complicated laws

Ohm's law in an anisotropic medium

The simple form of Ohm's Law stated above, in which an applied electric field $\underline{E}$ produces a current in the same direction, only holds for conducting media which are isotropic, that is, the same in all directions. This is certainly not the case in crystalline media, where the regular lattice will favour conduction in some directions more than in others.

The most general relation between $\underline{J}$ and $\underline{E}$ which is linear and is such that $\underline{J}$ vanishes when $\underline{E}$ vanishes is of the form

$$J_i = G_{ij} E_j$$

where G_{ij} are the components of the *conductivity tensor* in the chosen basis, and characterise the conduction properties when $\underline{J}$ and $\underline{E}$ are measured in that basis. Thus we need nine numbers, G_{ij}, to characterise the conductivity of an anisotropic medium. The conductivity tensor is an example of a **second rank tensor**.

Suppose we consider an orthogonal transformation of basis. Simply changing basis cannot alter the form of the physical law and so we conclude that

$$J'_i = G'_{ij} E'_j \quad \text{where} \quad J'_i = \lambda_{ij} J_j \quad \text{and} \quad E'_j = \lambda_{jk} E_k$$

Thus we deduce that

$$\lambda_{ij} J_j = \lambda_{ij} G_{jk} E_k = G'_{ij} \lambda_{jk} E_k$$

which we can rewrite as

$$(G'_{ij} \lambda_{jk} - \lambda_{ij} G_{jk}) E_k = 0$$

This must be true for arbitrary electric fields and hence

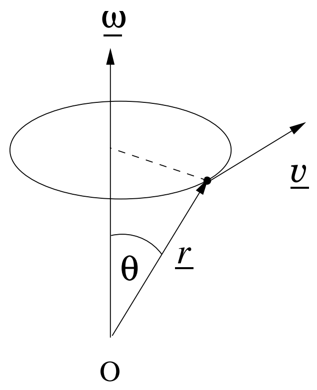
$$G'_{ij} \lambda_{jk} \;=\; \lambda_{ij} G_{jk}$$

Multiplying both sides by λ_{lk}, noting that $\lambda_{lk} \lambda_{jk} = \delta_{lj}$ and using the sifting property we find that

$$\boxed{G'_{il} = \lambda_{ij} \lambda_{lk} G_{jk}}$$

This exemplifies how the components of a second rank tensor change under an orthogonal transformation.

Rotating rigid body



Consider a particle of mass m at a point $\underline{r}$ in a rigid body rotating with **angular velocity** $\underline{\omega}$. Recall from Lecture 1 that $\underline{v} = \underline{\omega} \times \underline{r}$. You were asked to check that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r \sin\theta = \omega \times$ radius of circle in which the point is travelling

Now consider the **angular momentum** of the particle about the origin O, defined by $\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times (m\underline{v})$ where m is the mass of the particle.

Using the above expression for $\underline{v}$ we obtain

$$\underline{L} = m\underline{r} \times (\underline{\omega} \times \underline{r}) = m\left[\underline{\omega}(\underline{r} \cdot \underline{r}) - \underline{r}(\underline{r} \cdot \underline{\omega})\right] \tag{24}$$

where we have used the identity for the vector triple product. Note that only if $\underline{r}$ is perpendicular to $\underline{\omega}$ do we obtain $\underline{L} = mr^2\underline{\omega}$, which means that only then are $\underline{L}$ and $\underline{\omega}$ in the same direction.

Taking components of equation (24) in an orthonormal basis $\{\underline{e}_i\}$, we find that

$$\begin{aligned} L_i &= m\left[\omega_i(\underline{r} \cdot \underline{r}) - x_i(\underline{r} \cdot \underline{\omega})\right] \\ &= m\left[r^2\omega_i - x_i x_j \omega_j\right] \quad \text{noting that} \quad \underline{r} \cdot \underline{\omega} = x_j\omega_j \\ &= m\left[r^2\delta_{ij} - x_i x_j\right]\omega_j \quad \text{using} \quad \omega_i = \delta_{ij}\omega_j \end{aligned}$$

Thus

$$\boxed{L_i = I_{ij}(O)\,\omega_j \qquad \text{where} \qquad I_{ij}(O) = m\left[r^2\,\delta_{ij} - x_i x_j\right]}$$

$I_{ij}(O)$ are the components of the **inertia tensor**, relative to O, in the $\underline{e}_i$ basis. The inertia tensor is another example of a second rank tensor.

Summary of why we need tensors

(i) Physical laws often relate two vectors.

(ii) A second rank tensor provides a linear relation between two vectors which may be in different directions.

(iii) Tensors allow the generalisation of isotropic laws ('physics the same in all directions') to anisotropic laws ('physics different in different directions')

2.1.3 General properties

Scalars and vectors are called tensors of rank zero and one respectively, where *rank = no. of indices in a Cartesian basis.* We can also define tensors of rank greater than two.

The set of nine numbers, T_{ij}, representing a second rank tensor can be written as a 3×3 array

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

This of course is not true for higher rank tensors (which have more than 9 components).

We can rewrite the generic transformation law for a second rank tensor as follows:

$$T'_{ij} = \lambda_{ik}\lambda_{jl}\, T_{kl} = \lambda_{ik}\, T_{kl}\, (\lambda^T)_{lj}$$

Thus in matrix form the transformation law is

$$\boxed{T' = \lambda T \lambda^T}$$

Notes

(i) It is wrong to say that a second rank tensor is a matrix; rather the tensor is the fundamental object and is *represented in a given basis* by a matrix.

(ii) It is wrong to say a matrix is a tensor *e.g.* the transformation matrix λ is not a tensor but nine numbers defining the transformation between *two different bases.*

2.1.4 Invariants

Trace of a tensor: the trace of a tensor is defined as the sum of the diagonal elements T_{ii}. Consider the trace of the matrix representing the tensor in the transformed basis

$$\begin{aligned} T'_{ii} &= \lambda_{ir}\lambda_{is}T_{rs} \\ &= \delta_{rs}T_{rs} = T_{rr} \end{aligned}$$

Thus the trace is the same, evaluated in any basis and is a scalar invariant.

Determinant: it can be shown that the determinant is also an invariant.

Symmetry of a tensor: if the matrix T_{ij} representing the tensor is *symmetric* then

$$T_{ij} = T_{ji}$$

Under a change of basis

$$\begin{aligned} T'_{ij} &= \lambda_{ir}\lambda_{js}T_{rs} \\ &= \lambda_{ir}\lambda_{js}T_{sr} \quad \text{using symmetry} \\ &= \lambda_{is}\lambda_{jr}T_{rs} \quad \text{relabelling} \\ &= T'_{ji} \end{aligned}$$

Therefore a symmetric tensor remains symmetric under a change of basis. Similarly (exercise) an antisymmetric tensor $T_{ij} = -T_{ji}$ remains antisymmetric.

In fact one can decompose an arbitrary second rank tensor T_{ij} into a symmetric part S_{ij} and an antisymmetric part A_{ij} through

$$\boxed{S_{ij} = \frac{1}{2}\left[T_{ij} + T_{ji}\right] \quad A_{ij} = \frac{1}{2}\left[T_{ij} - T_{ji}\right]}$$

2.1.5 Eigenvectors

In general a second rank tensor maps a given vector onto a vector in a different direction: if a vector $\underline{n}$ has components n_i then

$$T_{ij}n_j = m_i ,$$

where m_i are components of $\underline{m}$, the vector that $\underline{n}$ is mapped onto.

However some special vectors called **eigenvectors** may exist such that $m_i = t\, n_i$ *i.e.* the new vector is in the *same* direction as the original vector. Eigenvectors usually have special physical significance (see later).

end of lecture 8

2.2 The Inertia Tensor

2.2.1 Computing the Inertia Tensor

We saw in the previous lecture that for a single particle of mass m, located at position $\underline{r}$ with respect to an origin O on the axis of rotation of a rigid body

$$\boxed{L_i = I_{ij}(O)\,\omega_j \qquad \text{where} \qquad I_{ij}(O) = m\left\{r^2\,\delta_{ij} - x_i x_j\right\}}$$

where $I_{ij}(O)$ are the components of the inertia tensor, relative to O, in the basis $\{\underline{e}_i\}$.

For a **collection of N particles** of mass m^α at $\underline{r}^\alpha$, where $\alpha = 1 \dots N$,

$$\boxed{I_{ij}(O) \;=\; \sum_{\alpha=1}^{N} m^\alpha \left\{ (\underline{r}^\alpha \cdot \underline{r}^\alpha)\, \delta_{ij} - x_i^\alpha x_j^\alpha \right\}} \tag{25}$$

For a **continuous body**, the sums become integrals, giving

$$\boxed{I_{ij}(O) \;=\; \int_V \rho(\underline{r}) \left\{ (\underline{r} \cdot \underline{r})\, \delta_{ij} - x_i x_j \right\} dV} \, .$$

Here, $\rho(\underline{r})$ is the density at position $\underline{r}$. $\rho(\underline{r})\, dV$ is the mass of the volume element dV at $\underline{r}$.

For laminae (flat objects) and solid bodies, these are 2- and 3-dimensional integrals respectively.

If the basis is *fixed relative to the body*, the $I_{ij}(O)$ are **constants** in time.

Consider the **diagonal** term

$$\begin{aligned} I_{11}(O) &= \sum_\alpha m^\alpha \left\{ (\underline{r}^\alpha \cdot \underline{r}^\alpha) - (x_1^\alpha)^2 \right\} \\ &= \sum_\alpha m^\alpha \left\{ (x_2^\alpha)^2 + (x_3^\alpha)^2 \right\} \\ &= \sum_\alpha m^\alpha\, (r_\perp^\alpha)^2 \, , \end{aligned}$$

where $r_\perp^\alpha$ is the **perpendicular distance** of m^α from the $\underline{e}_1$ axis through O.

This term is called the **moment of inertia** about the $\underline{e}_1$ axis. It is simply the mass of each particle in the body, multiplied by the square of its distance from the $\underline{e}_1$ axis, summed over all of the particles. Similarly the other diagonal terms are moments of inertia.

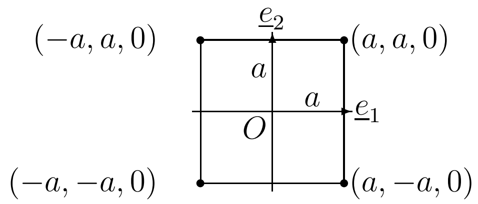
The **off-diagonal** terms are called the **products of inertia**, having the form, for example

$$I_{12}(O) \;=\; -\sum_\alpha m^\alpha\, x_1^\alpha\, x_2^\alpha \, .$$

Example

Consider 4 masses m at the vertices of a square of side $2a$.

(i) O **at centre of the square.**



For $m^{(1)} = m$ at $(a,a,0)$, $\underline{r}^{(1)} = a\underline{e}_1 + a\underline{e}_2$, so $\underline{r}^{(1)} \cdot \underline{r}^{(1)} = 2a^2$, $x_1^{(1)} = a$, $x_2^{(1)} = a$ and $x_3^{(1)} = 0$

$$I(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&2\end{pmatrix}.$$

For $m^{(2)} = m$ at $(a,-a,0)$, $\underline{r}^{(2)} = a\underline{e}_1 - a\underline{e}_2$, so $\underline{r}^{(2)} \cdot \underline{r}^{(2)} = 2a^2$, $x_1^{(2)} = a$ and $x_2^{(2)} = -a$

$$I(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}.$$

For $m^{(3)} = m$ at $(-a,-a,0)$, $\underline{r}^{(3)} = -a\underline{e}_1 - a\underline{e}_2$, so $\underline{r}^{(3)} \cdot \underline{r}^{(3)} = 2a^2$, $x_1^{(3)} = -a$ and $x_2^{(3)} = -a$

$$I(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&2\end{pmatrix}.$$

For $m^{(4)} = m$ at $(-a,a,0)$, $\underline{r}^{(4)} = -a\underline{e}_1 + a\underline{e}_2$, so $\underline{r}^{(4)} \cdot \underline{r}^{(4)} = 2a^2$, $x_1^{(4)} = -a$ and $x_2^{(4)} = a$

$$I(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}.$$

Adding up the four contributions gives the inertia tensor for all 4 particles as:–

$$\boxed{I(O) \;=\; 4ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}.}$$

Note that the final inertia tensor is diagonal and in this basis the products of inertia are all zero. (Of course there are other bases where the tensor is not diagonal.) This implies the basis vectors are eigenvectors of the inertia tensor. For example, if $\underline{\omega} = \omega(0,0,1)$ then $\underline{L}(O) = 8m\omega a^2(0,0,1)$.

In general $\underline{L}(O)$ is not parallel to $\underline{\omega}$. For example, if $\underline{\omega} = \omega(0,1,1)$ then $\underline{L}(O) = 4m\omega a^2(0,1,2)$. Note that the inertia tensors for the individual masses are not diagonal.

2.2.2 Two Useful Theorems

Perpendicular Axes Theorem

For a lamina, or collection of particles confined to a plane, (choose $\underline{e}_3$ as normal to the plane), with O in the plane

$$\boxed{I_{11}(O) + I_{22}(O) \;=\; I_{33}(O)}\,.$$

This is simply checked by using equation (25) on page 33 and noting $x_3^\alpha = 0$.

Parallel Axes Theorem

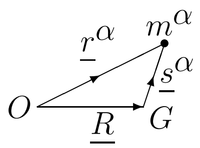
If G is the **centre of mass** of the body its position vector $\underline{R}$ is given by

$$\underline{R} = \sum_\alpha m^\alpha \underline{r}^\alpha \Big/ M,$$

where $\underline{r}^\alpha$ are the position vectors relative to O and $M = \sum_\alpha m^\alpha$, is the **total mass** of the system.

The parallel axes theorem states that

$$\boxed{I_{ij}(O) - I_{ij}(G) \;=\; M\left\{(\underline{R}\cdot\underline{R})\,\delta_{ij} - R_i R_j\right\}}\,,$$



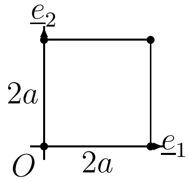
Proof: Let $\underline{s}^\alpha$ be the position of m^α **with respect to** G, then

$$\begin{aligned}
I_{ij}(G) &= \sum_\alpha m^\alpha \left\{(\underline{s}^\alpha\cdot\underline{s}^\alpha)\,\delta_{ij} - s_i^\alpha s_j^\alpha\right\}\;; \\
I_{ij}(O) &= \sum_\alpha m^\alpha \left\{(\underline{r}^\alpha\cdot\underline{r}^\alpha)\,\delta_{ij} - x_i^\alpha x_j^\alpha\right\} \\
&= \sum_\alpha m^\alpha \left\{(\underline{R}+\underline{s}^\alpha)^2\,\delta_{ij} - (\underline{R}+\underline{s}^\alpha)_i(\underline{R}+\underline{s}^\alpha)_j\right\} \\
&= M\left\{R^2\delta_{ij} - R_i R_j\right\} + \sum_\alpha m^\alpha \left\{(\underline{s}^\alpha\cdot\underline{s}^\alpha)\,\delta_{ij} - s_i^\alpha s_j^\alpha\right\} \\
&\qquad +2\,\delta_{ij}\,\underline{R}\cdot\sum_\alpha m^\alpha \underline{s}^\alpha - R_i\sum_\alpha m^\alpha s_j^\alpha - R_j\sum_\alpha m^\alpha s_i^\alpha \\
&= M\left\{R^2\delta_{ij} - R_i R_j\right\} + I_{ij}(G)
\end{aligned}$$

the cross terms vanishing since

$$\sum_\alpha m^\alpha s_i^\alpha \;=\; \sum_\alpha m^\alpha \left(r_i^\alpha - R_i\right) \;=\; 0\,.$$

Example of use of Parallel Axes Theorem.



Consider the same arrangement of masses as before but with O at one corner of the square *i.e.* a (massless) lamina of side $2a$, with masses m at each corner and the origin O at the bottom, left so that the masses are at $(0,0,0)$, $(2a,0,0)$, $(0,2a,0)$ and $(2a,2a,0)$

We have $M = 4m$ and

$$\begin{aligned}\underline{OG} \;=\; \underline{R} \;&=\; \frac{1}{4m}\left\{m(0,0,0) + m(2a,0,0) + m(0,2a,0) + m(2a,2a,0)\right\}\\ &=\; (a,a,0)\end{aligned}$$

and so G is at the centre of the square and $R^2 = 2a^2$. We can now use the parallel axis theorem to relate the inertia tensor of the previous example to that of the present

$$I(O) - I(G) \;=\; 4m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; 4ma^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&2\end{pmatrix}.$$

From the previous example,

$$I(G) \;=\; 4ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix} \quad \text{and hence}$$

$$I(O) \;=\; 4ma^2\begin{pmatrix}1+1&0-1&0\\0-1&1+1&0\\0&0&2+2\end{pmatrix} \;=\; 4ma^2\begin{pmatrix}2&-1&0\\-1&2&0\\0&0&4\end{pmatrix}$$

end of lecture 9

2.3 Eigenvectors of Real, Symmetric Tensors

If T is a (2nd-rank) tensor an eigenvector $\underline{n}$ of T obeys (in any basis)

$$\boxed{T_{ij} n_j = t\; n_i}\;.$$

where t is the eigenvalue of the eigenvector.

The tensor acts on the eigenvector to produce a vector in the *same* direction.

The direction of $\underline{n}$ doesn't depend on the basis although its components do (because $\underline{n}$ is a vector) and is sometimes referred to as a **principal axis**; t is a scalar (doesn't depend on basis) and is sometimes referred to as a **principal value**.

2.3.1 Construction of the Eigenvectors

Since $n_i = \delta_{ij} n_j$, we can write the equation for an eigenvector as

$$\left(T_{ij} - t\,\delta_{ij}\right) n_j \;=\; 0\,.$$

This set of three linear equations has a non-trivial solution (i.e. a solution $\underline{n} \neq 0$) iff

$$\boxed{\det\left(T - t\,\mathbb{1}\right) \;\equiv\; 0}\,.$$

i.e.

$$\begin{vmatrix} T_{11} - t & T_{12} & T_{13} \\ T_{21} & T_{22} - t & T_{23} \\ T_{31} & T_{32} & T_{33} - t \end{vmatrix} \;=\; 0\,.$$

This is equation, known as the 'characteristic' or 'secular' equation, is a **cubic** in t, giving 3 real solutions $t^{(1)}$, $t^{(2)}$ and $t^{(3)}$ and corresponding eigenvectors $\underline{n}^{(1)}$, $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$.

Example:

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The characteristic equation reads

$$\begin{vmatrix} 1-t & 1 & 0 \\ 1 & -t & 1 \\ 0 & 1 & 1-t \end{vmatrix} \;=\; 0.$$

Thus

$$(1-t)\{t(t-1) - 1\} - \{(1-t) - 0\} \;=\; 0$$

and so

$$(1-t)\{t^2 - t - 2\} \;=\; (1-t)(t-2)(t+1) \;=\; 0.$$

Thus the solutions are $t = 1$, $t = 2$ and $t = -1$.

Check: The sum of the eigenvalues is 2, and is equal to the **trace** of the tensor; the reason for this will become apparent next lecture.

We now find the eigenvector for each of these eigenvalues, by solving $T_{ij}\,n_j = tn_i$

$$\begin{array}{rcrcrcl} (1-t)\,n_1 & + & n_2 & & & = & 0 \\ n_1 & - & t\,n_2 & + & n_3 & = & 0 \\ & & n_2 & + & (1-t)\,n_3 & = & 0. \end{array}$$

for $t = 1$, $t = 2$ and $t = -1$ in turn.

For $t = t^{(1)} = 1$, we denote the corresponding eigenvector by $\underline{n}^{(1)}$ and the equations for the components of $\underline{n}^{(1)}$ are (dropping the label (1)):

$$\left.\begin{array}{rcrcrcl} & & n_2 & & & = & 0 \\ n_1 & - & n_2 & + & n_3 & = & 0 \\ & & n_2 & & & = & 0 \end{array}\right\} \Longrightarrow n_2 \;=\; 0\,;n_3 \;=\; -n_1\,.$$

Thus $n_1 : n_2 : n_3 \;=\; 1 : 0 : -1$ and a *unit* vector in the direction of $\underline{n}^{(1)}$ is

$$\underline{\hat{n}}^{(1)} \;=\; \frac{1}{\sqrt{2}}(1, 0, -1)\;.$$

[Note that we could equally well have chosen $\underline{\hat{n}}^{(1)} \;=\; \frac{-1}{\sqrt{2}}(1, 0, -1)\,.$]

For $t = t^{(2)} = 2$, the equations for the components of $\underline{n}^{(2)}$ are:

$$\left.\begin{array}{rcrcrcl} -n_1 & + & n_2 & & & = & 0 \\ n_1 & - & 2n_2 & + & n_3 & = & 0 \\ & & n_2 & - & n_3 & = & 0 \end{array}\right\} \Longrightarrow n_2 \;=\; n_3 \;=\; n_1\,.$$

Thus $n_1 : n_2 : n_3 \;=\; 1 : 1 : 1$ and a *unit* vector in the direction of $\underline{n}^{(2)}$ is

$$\underline{\hat{n}}^{(2)} \;=\; \frac{1}{\sqrt{3}}(1, 1, 1)\;.$$

For $t = t^{(3)} = -1$, a similar calculation (exercise) gives

$$\underline{\hat{n}}^{(3)} \;=\; \frac{1}{\sqrt{6}}(1, -2, 1)\;.$$

Note that $\underline{\hat{n}}^{(1)} \cdot \underline{\hat{n}}^{(2)} \;=\; \underline{\hat{n}}^{(1)} \cdot \underline{\hat{n}}^{(3)} \;=\; \underline{\hat{n}}^{(2)} \cdot \underline{\hat{n}}^{(3)} \;=\; 0$ and so the eigenvectors are mutually orthogonal.

The scalar triple product of the triad $\underline{\hat{n}}^{(1)}$, $\underline{\hat{n}}^{(2)}$ and $\underline{\hat{n}}^{(3)}$, with the above choice of signs, is -1, and so they form a *left-handed* basis. Changing the sign of *one* (or all three) of the vectors would produce a right-handed basis.

2.3.2 Important Theorem and Proof

Theorem: If T_{ij} is *real* and *symmetric*, its eigenvalues are *real*. The eigenvectors corresponding to *distinct* eigenvalues are *orthogonal*.

Proof: Let $\underline{a}$ and $\underline{b}$ be eigenvectors, with eigenvalues t^a and t^b respectively, then:–

$$T_{ij}a_j \;=\; t^a\, a_i \tag{26}$$

$$T_{ij}b_j \;=\; t^b\, b_i \tag{27}$$

We multiply equation (26) by b_i^*, and sum over i, giving:–

$$T_{ij} a_j b_i^* \;=\; t^a \, a_i b_i^* \tag{28}$$

We now take the complex conjugate of equation (27), multiply by a_i and sum over i, to give:–

$$T_{ij}^* b_j^* a_i \;=\; t^{b*} \, b_i^* a_i \tag{29}$$

Since T_{ij} is *real* and *symmetric*, $T_{ij}^* = T_{ji}$, and so:–

$$\begin{aligned} \text{l.h. side of equation (29)} \;&=\; T_{ji} b_j^* a_i \\ &=\; T_{ij} b_i^* a_j \;=\; \text{l.h. side of equation (28).} \end{aligned}$$

Subtracting (29) from (28) gives:–

$$\boxed{\left(t^a - t^{b*}\right) a_i b_i^* \;=\; 0} \; .$$

Case 1: consider what happens if $\underline{b} = \underline{a}$,

$$a_i a_i^* \;=\; \sum_{i=1}^{3} |a_i|^2 \;>\; 0 \;\; \text{for all non-zero } \underline{a},$$

and so

$$\boxed{t^a \;=\; t^{a*}} \; .$$

Thus, we have shown that the eigenvalues are real.

Since t is real and T_{ij} are real, real $\underline{a}, \underline{b}$ can be found.

Case 2: now consider $\underline{a} \neq \underline{b}$, in which case $t^a \neq t^b$ by hypothesis:

$$\left(t^a - t^b\right) a_i b_i \;=\; 0 \; .$$

If $t^a \neq t^b$, then $a_i b_i = 0$, implying

$$\boxed{\underline{a} \cdot \underline{b} \;=\; 0} \; .$$

Thus the eigenvectors are orthogonal if the eigenvalues are distinct.

2.3.3 Degenerate eigenvalues

If the characteristic equation is of the form

$$(t^{(1)} - t)(t^{(2)} - t)^2 = 0$$

there is a repeated root and we have a *doubly degenerate* eigenvalue $t^{(2)}$.

Claim: In the case of a real, symmetric tensor we can nevertheless always find TWO mutually orthogonal solutions for $\underline{n}^{(2)}$ (which are both orthogonal to $\underline{n}^{(1)}$).

Example

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow |T - t\mathbb{1}| = \begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix} = 0 \Rightarrow \boxed{t = 2 \text{ and } t = -1 \text{ (twice)}} \,.$$

For $t = t^{(1)} = 2$ with eigenvector $\underline{n}^{(1)}$

$$\left.\begin{array}{rcrcrcl} -2n_1 & + & n_2 & + & n_3 & = & 0 \\ n_1 & - & 2n_2 & + & n_3 & = & 0 \\ n_1 & + & n_2 & - & 2n_3 & = & 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} n_2 & = & n_3 = n_1 \\ \underline{\hat{n}}^{(1)} & = & \frac{1}{\sqrt{3}}(1,1,1) \,. \end{array}\right.$$

For $t = t^{(2)} = -1$ with eigenvector $\underline{n}^{(2)}$

$$n_1^{(2)} + n_2^{(2)} + n_3^{(2)} = 0$$

is the only independent equation. This can be written as $\underline{n}^{(1)} \cdot \underline{n}^{(2)} = 0$ which is the equation for a plane normal to $\underline{n}^{(1)}$. Thus any vector orthogonal to $\underline{n}^{(1)}$ is an eigenvector with eigenvalue -1.

If we choose $n_3^{(2)} = 0$, then $n_2^{(2)} = -n_1^{(2)}$ and a possible unit eigenvector is

$$\underline{\hat{n}}^{(2)} = \frac{1}{\sqrt{2}}(1, -1, 0) \,.$$

If we require the third eigenvector $\underline{n}^{(3)}$ to be orthogonal to $\underline{n}^{(2)}$, then we must have $n_2^{(3)} = n_1^{(3)}$. The equations then give $n_3^{(3)} = -2n_1^{(3)}$ and so

$$\underline{\hat{n}}^{(3)} = \frac{1}{\sqrt{6}}(1, 1, -2) \,.$$

Alternatively, the third eigenvector can be calculated by using $\underline{\hat{n}}^{(3)} = \pm\underline{\hat{n}}^{(1)} \times \underline{\hat{n}}^{(2)}$, the sign chosen determining the handedness of the triad $\underline{\hat{n}}^{(1)}$, $\underline{\hat{n}}^{(2)}$, $\underline{\hat{n}}^{(3)}$. This particular pair, $\underline{n}^{(2)}$ and

$\underline{n}^{(3)}$, is just one of an *infinite number* of orthogonal pairs that are eigenvectors of T_{ij} — all lying in the plane normal to $\underline{n}^{(1)}$.

If the characteristic equation is of form

$$(t^{(1)} - t)^3 = 0$$

then we have a triply degenerate eigenvalue $t^{(1)}$. In fact, this only occurs if the tensor is equal to $t^{(1)}\delta_{ij}$ which means it is 'isotropic' and any direction is an eigenvector with eigenvalue $t^{(1)}$.

end of lecture 10

2.4 Diagonalisation of a Real, Symmetric Tensor

In the basis $\{\underline{e}_i\}$ the tensor T_{ij} is, in general, non-diagonal. i.e. T_{ij} is non-zero for $i \neq j$. However if we transform to a basis constructed from the normalised eigenvectors—the 'principal axes'—we find that the tensor becomes diagonal.

Transform to the basis $\{\underline{e}_i{}'\}$ chosen such that

$$\boxed{\underline{e}_i{}' \;=\; \underline{n}^{(i)}}\;,$$

where $\underline{n}^{(i)}$ are the three *normalized*, and *orthogonal*, eigenvectors of T_{ij} with eigenvalues $t^{(i)}$ respectively.

Now

$$\lambda_{ij} \;=\; \underline{e}_i{}' \cdot \underline{e}_j \;=\; \underline{n}^{(i)} \cdot \underline{e}_j \;=\; n_j^{(i)}\;.$$

i.e. the **rows** of λ are the components of the **normalised eigenvectors** of T.

In the basis $\{\underline{e}_i{}'\}$

$$T'_{ij} = (\lambda T \lambda^T)_{ij}$$

Now since the **columns** of λ^T are the **normalised eigenvectors** of T we see that

$$T\lambda^T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix} = \begin{pmatrix} t^{(1)}n_1^{(1)} & t^{(2)}n_1^{(2)} & t^{(3)}n_1^{(3)} \\ t^{(1)}n_2^{(1)} & t^{(2)}n_2^{(2)} & t^{(3)}n_2^{(3)} \\ t^{(1)}n_3^{(1)} & t^{(2)}n_3^{(2)} & t^{(3)}n_3^{(3)} \end{pmatrix}$$

$$\lambda T\lambda^T \;=\; \begin{pmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{pmatrix} \begin{pmatrix} t^{(1)}n_1^{(1)} & t^{(2)}n_1^{(2)} & t^{(3)}n_1^{(3)} \\ t^{(1)}n_2^{(1)} & t^{(2)}n_2^{(2)} & t^{(3)}n_2^{(3)} \\ t^{(1)}n_3^{(1)} & t^{(2)}n_3^{(2)} & t^{(3)}n_3^{(3)} \end{pmatrix} = \begin{pmatrix} t^{(1)} & 0 & 0 \\ 0 & t^{(2)} & 0 \\ 0 & 0 & t^{(3)} \end{pmatrix}$$

from the orthonormality of the $\underline{n}^{(i)}$ (rows of λ; columns of λ^T).

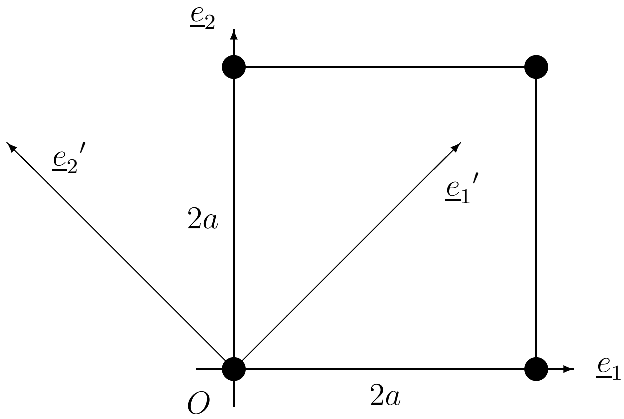
Thus, with respect to a basis defined by the eigenvectors or principal axes of the tensor, the tensor has *diagonal form*. [i.e. $T' = \text{diag}\{t^{(1)}, t^{(2)}, t^{(3)}\}$.] The diagonal basis is often referred to as the '**principal axes basis**'.

Note: In the diagonal basis the trace of a tensor is the sum of the eigenvalues; the determinant of the tensor is the product of the eigenvalues. Since the trace and determinant are invariants this means that in any basis the trace and determinant are the sum and products of the eigenvalues respectively.

Example: Diagonalisation of Inertia Tensor. Consider the inertia tensor for four masses arranged in a square with the origin at the left hand corner (see lecture 9 p 36):

$$I(O) \quad = \quad 4ma^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

It is easy to check (exercise) that the eigenvectors (or principal axes of inertia) are $(\underline{e}_1 + \underline{e}_2)$ (eigenvalue $4ma^2$), $(\underline{e}_1 - \underline{e}_2)$ (eigenvalue $12ma^2$) and $\underline{e}_3$ (eigenvalue $16ma^2$).



Defining the $\underline{e}_i{}'$ basis as normalised eigenvectors: $\underline{e}_1{}' = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)$; $\underline{e}_2{}' = \frac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_2)$; $\underline{e}_3{}' = \underline{e}_3$, one obtains

$$\lambda \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (\sim \text{rotation of } \pi/4 \text{ about } \underline{e}_3 \text{ axis})$$

and the inertia tensor in the basis $\{\underline{e}_i{}'\}$ has components $I'_{ij}(O) = \left(\lambda\, I(O)\lambda^T\right)_{ij}$ so that

$$
\begin{aligned}
I'(O) &= 4ma^2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= 4ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} .
\end{aligned}
$$

We see that the tensor is diagonal with diagonal elements which are the eigenvalues (principal moments of inertia).

Remark: Diagonalisability is a very special and useful property of real, symmetric tensors. It is a property also shared by the more general class of Hermitean operators which you will meet in quantum mechanics in third year. A general tensor does not share the property. For example a real non-symmetric tensor cannot be diagonalised.

2.4.1 Symmetry and Eigenvectors of the Inertia Tensor

In the previous example the eigenvectors had some physical significance: in the original basis $\underline{e}_3$ is perpendicular to the plane where the masses lie; $\underline{e}_1 + \underline{e}_2$ is along the diagonal of the square.

By using the transformation law for the inertia tensor we can see how the symmetry of the mass arrangement is related to the eigenvectors of the tensor. First we need to define symmetry axes and planes.

A **Symmetry Plane** is a plane under reflection in which the distribution of mass remains unchanged *e.g.* for a lamina with normal $\underline{e}_3$ the $\underline{e}_1 - \underline{e}_2$ plane is a reflection symmetry plane.

Claim: A normal to a symmetry plane is an eigenvector

Proof: Choose $\underline{e}_3$ as the normal. Now since the mass distribution is invariant under reflection in the symmetry plane, the representation of the tensor must be unchanged when the axes are reflected in the plane *i.e.* the tensor should look exactly the same when the axes have been transformed in such a way that the mass distribution with repect to the new axes is the same as the mass distribution with respect to the old axes.

$$
\therefore \quad I' = \lambda I \lambda^T = I \quad \text{for } \lambda \text{ a reflection in the } \underline{e}_1\text{–}\underline{e}_2 \text{ plane} \quad \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .
$$

Calculating $I' = \lambda I \lambda^T$ gives

$$I' = \begin{pmatrix} I_{11} & I_{12} & -I_{13} \\ I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix} = I \quad \Rightarrow \quad I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

and $\underline{e}_3$ is an eigenvector, eigenvalue I_{33}.

An **m-fold Symmetry Axis** is an axis about which rotation of the system by $2\pi/m$ leaves the mass distribution unchanged *e.g.* for the example of the previous subsection the diagonal of the square is 2-fold symmetry axis.

Claim: A 2-fold symmetry axis is an eigenvector

Proof: Choose $\underline{e}_3$ as the symmetry axis. Now since the mass distribution is invariant under rotation of π about this axis, the representation of the tensor must be unchanged when the axes are rotated by π about $\underline{e}_3$

$$\therefore \quad I' = \lambda I \lambda^T = I \quad \text{for } \lambda \text{ a rotation of } \pi \text{ about } \underline{e}_3 \qquad \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculating $I' = \lambda I \lambda^T$ gives

$$I' = \begin{pmatrix} I_{11} & I_{12} & -I_{13} \\ I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix} = I \quad \Rightarrow \quad I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

and $\underline{e}_3$ is an eigenvector, eigenvalue I_{33}.

Claim: An m-fold symmetry axis is an eigenvector and for $m > 2$ the orthogonal plane is a degenerate eigenspace *i.e.* if $\underline{e}_3$ is chosen as the symmetry axis then I is of the form

$$I = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}$$

See *e.g.* the example in lecture 9 p.34/35 which has $\underline{e}_3$ as a 4-fold symmetry axis.

Proof: The idea is the same as the $m = 2$ case above. Because it is a bit more complicated we do not include it here.

Note: The limit $m \to \infty$ yields a continuous symmetry axis. *e.g.* a cylinder, a cone ..

2.4.2 Summary

a. *The most general body with no symmetry:*

the 3 orthogonal eigenvectors have to be found the hard way!

b. *A body with a symmetry plane:*

the normal to the symmetry plane is an eigenvector.

c. *A body with a 2-fold symmetry axis:*

the symmetry axis is an eigenvector.

d. *A body with an m-fold symmetry axis ($m > 2$):*

the symmetry axis is an eigenvector; there are degenerate eigenvectors normal to the symmetry axis.

e. *A body with spherical symmetry:*

any vector is an eigenvector with the same eigenvalue! (triple degeneracy)

end of lecture 11

3 Fields

3.1 Examples of Fields

In physics we often have to consider properties that vary in some region of space e.g. temperature of a body. To do this we require the concept of fields.

If to each point $\underline{r}$ in some region of ordinary 3-d space there corresponds a **scalar** $\phi(x_1, x_2, x_3)$, then $\phi(\underline{r})$ is a **scalar field**.

Examples: temperature distribution in a body $T(\underline{r})$, pressure in the atmosphere $P(\underline{r})$, electric charge density or mass density $\rho(\underline{r})$, electrostatic potential $\phi(\underline{r})$.

Similarly a **vector field** assigns a vector $\underline{V}(x_1, x_2, x_3)$ to each point $\underline{r}$ of some region.

Examples: velocity in a fluid $\underline{v}(\underline{r})$, electric current density $\underline{J}(\underline{r})$, electric field $\underline{E}(\underline{r})$, magnetic field $\underline{B}(\underline{r})$

A vector field in 2-d can be represented graphically, at a carefully selected set of points $\underline{r}$, by an arrow whose length and direction is proportional to $\underline{V}(\underline{r})$ *e.g.* wind velocity on a weather forecast chart.

3.1.1 Level Surfaces of a Scalar Field

If $\phi(\underline{r})$ is a non-constant scalar field, then the equation $\phi(\underline{r}) = c$ where c is a constant, defines a **level surface** (or equipotential) of the field. Level surfaces do not intersect (else ϕ would be multi-valued at the point of intersection).

Familiar examples in two dimensions, where they are level curves rather than level surfaces, are the contours of constant height on a geographical map, $h(x_1, x_2) = c$. Also isobars on a weather map are level curves of pressure $P(x_1, x_2) = c$.

Examples in three dimensions:

(i) Suppose that

$$\phi(\underline{r}) = x_1^2 + x_2^2 + x_3^2 = x^2 + y^2 + z^2$$

The level surface $\phi(\underline{r}) = c$ is a sphere of radius $\sqrt{c}$ centred on the origin. As c is varied, we obtain a family of level surfaces which are concentric spheres.

(ii) Electrostatic potential due to a point charge q situated at the point $\underline{a}$ is

$$\phi(\underline{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\underline{r} - \underline{a}|}$$

The level surfaces are concentric spheres centred on the point $\underline{a}$.

(iii) Let $\phi(\underline{r}) = \underline{k} \cdot \underline{r}$. The level surfaces are planes $\underline{k} \cdot \underline{r} =$ constant with normal $\underline{k}$.

(iv) Let $\phi(\underline{r}) = \exp(i\underline{k} \cdot \underline{r})$. Note that this a complex scalar field. Since $\underline{k} \cdot \underline{r} =$ constant is the equation for a plane, the level surfaces are planes.

3.1.2 Gradient of a Scalar Field

How does a scalar field change as we change position?

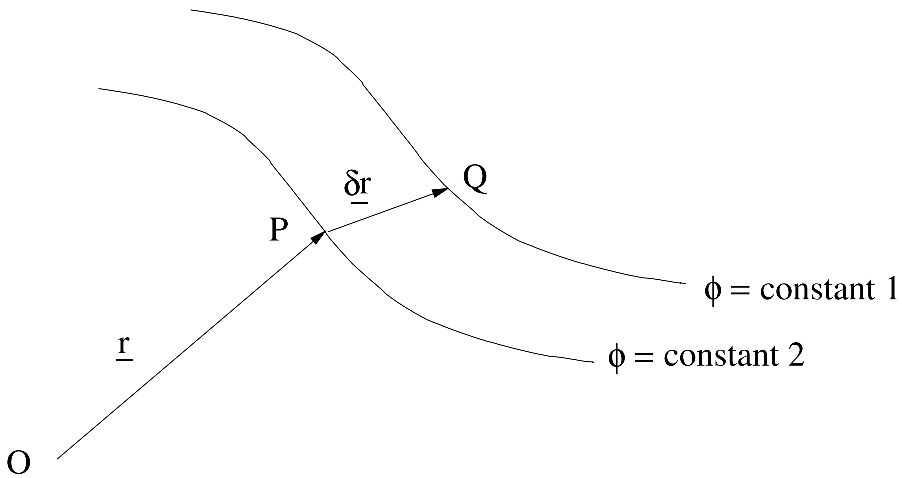
As an example think of a 2-d contour map of the height $h = h(x, y)$ of a hill say. The height is a scalar field. If we are on the hill and move in the $x - y$ plane then the change in height will depend on the direction in which we move (unless the hill is completely flat!). For example there will be a direction in which the height increases most steeply ('straight up the hill') We now introduce a formalism to describe how a scalar field $\phi(\underline{r})$ changes as a function of $\underline{r}$.

Mathematical Note: A scalar field $\phi(\underline{r}) = \phi(x_1, x_2, x_3)$ is said to be *continuously differentiable* in a region R if its first order partial derivatives

$$\frac{\partial \phi(\underline{r})}{\partial x_1}, \quad \frac{\partial \phi(\underline{r})}{\partial x_2} \quad \text{and} \quad \frac{\partial \phi(\underline{r})}{\partial x_3}$$

exist and are *continuous* at every point $\underline{r} \in R$. We will generally assume scalar fields are continuously differentiable.

Let $\phi(\underline{r})$ be a scalar field. Consider 2 nearby points: P (position vector $\underline{r}$) and Q (position vector $\underline{r} + \underline{\delta r}$). Assume P and Q lie on *different* level surfaces as shown:



Now use Taylor's theorem for a function of 3 variables to evaluate the change in ϕ as we move from P to Q

$$\begin{aligned}
\delta\phi &\equiv \phi(\underline{r} + \underline{\delta r}) - \phi(\underline{r}) \\
&= \phi(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3) - \phi(x_1, x_2, x_3) \\
&= \frac{\partial \phi(\underline{r})}{\partial x_1}\,\delta x_1 + \frac{\partial \phi(\underline{r})}{\partial x_2}\,\delta x_2 + \frac{\partial \phi(\underline{r})}{\partial x_3}\,\delta x_3 + O(\,\delta x_i^2\,)
\end{aligned}$$

where we have assumed that the higher order partial derivatives exist. Neglecting terms of order $(\,\delta x_i^2\,)$ we can write

$$\boxed{\delta\phi = \underline{\nabla}\;\phi(\underline{r})\cdot\underline{\delta r}}$$

where the 3 quantities

$$\left(\underline{\nabla}\;\phi(\underline{r})\right)_i = \frac{\partial\phi(\underline{r})}{\partial x_i}$$

form the Cartesian components of a **vector field**. We write

$$\boxed{\underline{\nabla}\;\phi(\underline{r}) \equiv \underline{e}_i\,\frac{\partial\phi(\underline{r})}{\partial x_i} = \underline{e}_1\,\frac{\partial\phi(\underline{r})}{\partial x_1} + \underline{e}_2\,\frac{\partial\phi(\underline{r})}{\partial x_2} + \underline{e}_3\,\frac{\partial\phi(\underline{r})}{\partial x_3}}$$

or in the old 'x, y, z' notation (where $x_1 = x$, $x_2 = y$ and $x_3 = z$)

$$\underline{\nabla}\;\phi(\underline{r}) = \underline{e}_1\,\frac{\partial\phi(\underline{r})}{\partial x} + \underline{e}_2\,\frac{\partial\phi(\underline{r})}{\partial y} + \underline{e}_3\,\frac{\partial\phi(\underline{r})}{\partial z}$$

The **vector field** $\underline{\nabla}\,\phi(\underline{r})$, pronounced "grad phi", is called the **gradient** of $\phi(\underline{r})$.

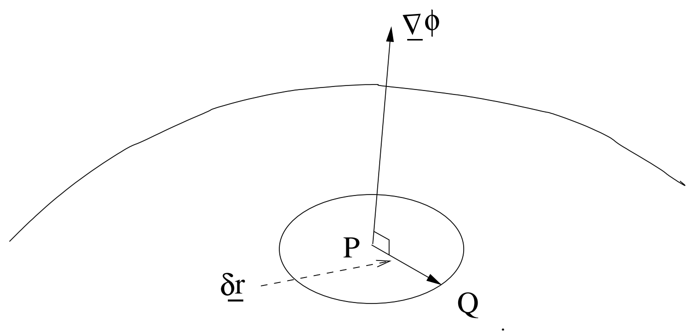
Example: calculate the gradient of $\phi = r^2 = x^2 + y^2 + z^2$

$$\begin{aligned}\underline{\nabla}\phi(\underline{r}) &= \left(\underline{e}_1\,\frac{\partial}{\partial x} + \underline{e}_2\,\frac{\partial}{\partial y} + \underline{e}_3\,\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)\\ &= 2x\,\underline{e}_1 + 2y\,\underline{e}_2 + 2z\,\underline{e}_3 = 2\underline{r}\end{aligned}$$

3.1.3 Interpretation of the gradient

In deriving the expression for $\delta\phi$ above, we assumed that the points P and Q lie on *different* level surfaces. Now consider the situation where P and Q are nearby points on the *same* level surface. In that case $\delta\phi = 0$ and so

$$\delta\phi = \underline{\nabla}\phi(\underline{r})\cdot\underline{\delta r} = 0$$



The infinitesimal vector $\underline{\delta r}$ lies in the level surface at $\underline{r}$, and the above equation holds for all such $\underline{\delta r}$, hence

$\underline{\nabla}\phi(\underline{r})$ is **normal to the level surface at** $\underline{r}$.

To construct a **unit normal** $\underline{\hat{n}}(\underline{r})$ to the level surface at $\underline{r}$, we divide $\underline{\nabla}\phi(\underline{r})$ by its length

$$\underline{\hat{n}}(\underline{r}) = \frac{\underline{\nabla}\phi(\underline{r})}{|\underline{\nabla}\phi(\underline{r})|} \qquad \big(\text{valid for} \quad |\underline{\nabla}\phi(\underline{r})| \neq 0\big)$$

3.1.4 Directional Derivative

Now consider the change, $\delta\phi$, produced in ϕ by moving distance δs in some direction say $\underline{\hat{s}}$. Then $\underline{\delta r} = \underline{\hat{s}}\delta s$ and

$$\delta\phi = \underline{\nabla}\phi(\underline{r}) \cdot \underline{\delta r} = (\underline{\nabla}\phi(\underline{r}) \cdot \underline{\hat{s}})\ \delta s$$

As $\delta s \to 0$, the rate of change of ϕ as we move in the direction of $\underline{\hat{s}}$ is

$$\frac{d\phi(\underline{r})}{ds} = \underline{\hat{s}} \cdot \underline{\nabla}\phi(\underline{r}) = |\underline{\nabla}\phi(\underline{r})|\ \cos\theta \tag{30}$$

where θ is the angle between $\underline{\hat{s}}$ and the normal to the level surface at $\underline{r}$.

$\underline{\hat{s}} \cdot \underline{\nabla}\phi(\underline{r})$ is the **directional derivative** of the scalar field ϕ in the direction of $\underline{\hat{s}}$.

Note that the directional derivative has its *maximum* value when $\underline{s}$ is parallel to $\underline{\nabla}\phi(\underline{r})$, and is *zero* when $\underline{s}$ lies in the level surface. Therefore

$\underline{\nabla}\phi$ points in the direction of the **maximum** rate of increase in ϕ

Also recall that this direction is normal to the level surface. For a familiar example think of the contour lines on a map. The steepest direction is perpendicular to the contour lines.

Example: Find the directional derivative of $\phi = xy(x+z)$ at point $(1, 2, -1)$ in the $(\underline{e}_1 + \underline{e}_2)/\sqrt{2}$ direction.

$$\underline{\nabla}\phi = (2xy + yz)\underline{e}_1 + x(x+z)\underline{e}_2 + xy\underline{e}_3 = 2\underline{e}_1 + 2\underline{e}_3$$

at (1,2,-1). Thus at this point

$$\frac{1}{\sqrt{2}}\,(\underline{e}_1 + \underline{e}_2) \cdot \underline{\nabla}\phi = \sqrt{2}$$

Physical example: Let $T(\underline{r})$ be the temperature of the atmosphere at the point $\underline{r}$. An object flies through the atmosphere with velocity $\underline{v}$. Obtain an expression for the rate of change of temperature experienced by the object.

As the object moves from $\underline{r}$ to $\underline{r} + \underline{\delta r}$ in time δt, it sees a change in temperature

$$\delta T(\underline{r}) \;=\; \underline{\nabla} T(\underline{r}) \cdot \underline{\delta r} \;=\; \left(\underline{\nabla} T(\underline{r}) \cdot \frac{\underline{\delta r}}{\delta t} \right) \; \delta t$$

Taking the limit $\delta t \to 0$ we obtain

$$\frac{dT(\underline{r})}{dt} = \underline{v} \cdot \underline{\nabla} T(\underline{r})$$

end of lecture 12

3.2 More on Gradient; the Operator 'Del'

3.2.1 Examples of the Gradient in Physical Laws

Gravitational force due to Earth: Consider the potential energy of a particle of mass m at a height z above the Earth's surface $V = mgz$. Then the force due to gravity can be written as

$$\underline{F} = -\underline{\nabla}\, V = -mg\; \underline{e}_3$$

Newton's Law of Gravitation: Now consider the gravitational force on a mass m at $\underline{r}$ due to a mass m_0 at the origin We can write this as

$$\underline{F} \;=\; -\frac{Gmm_0}{r^2}\,\underline{\hat{r}} \;=\; -\underline{\nabla}\, V$$

where the potential energy $V = -Gmm_0/r$ (see p.51 for how to calculate $\underline{\nabla}(1/r)$).

In these two examples we see that the force acts down the potential energy gradient.

3.2.2 Examples on gradient

Last lecture some examples using 'xyz' notation were given. Here we do some exercises with suffix notation. As usual suffix notation is most convenient for proving more complicated identities.

1. Let $\phi(\underline{r}) = r^2 = x_1^2 + x_2^2 + x_3^2$, then

$$\underline{\nabla}\phi(\underline{r}) = \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right)(x_1^2 + x_2^2 + x_3^2) = 2x_1\,\underline{e}_1 + 2x_2\,\underline{e}_2 + 2x_3\,\underline{e}_3 = 2\underline{r}$$

In suffix notation

$$\underline{\nabla}\phi(\underline{r}) = \underline{\nabla}\, r^2 = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right)(x_j x_j) = \underline{e}_i\,(\delta_{ij}x_j + x_j\delta_{ij}) = \underline{e}_i\,2x_i = 2\underline{r}$$

In the above we have used the important property of partial derivatives

$$\boxed{\frac{\partial x_i}{\partial x_j} = \delta_{ij}}$$

The level surfaces of r^2 are spheres centred on the origin, and the gradient of r^2 at $\underline{r}$ points radially outward with magnitude $2r$.

2. Let $\phi = \underline{a}\cdot\underline{r}$ where $\underline{a}$ is a *constant* vector.

$$\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right)(a_j x_j) = \underline{e}_i\,a_j\delta_{ij} = \underline{a}$$

This is not surprising, since the level surfaces $\underline{a}\cdot\underline{r} = c$ are planes orthogonal to $\underline{a}$.

3. Let $\phi(\underline{r}) = r = \sqrt{x_1^2 + x_2^2 + x_3^2} = (x_j x_j)^{1/2}$

$$\begin{aligned}
\underline{\nabla}\, r &= \left(\underline{e}_i \frac{\partial}{\partial x_i}\right)(x_j x_j)^{1/2} \\
&= \underline{e}_i\,\frac{1}{2}\,(x_j x_j)^{-1/2}\,\frac{\partial}{\partial x_i}(x_k x_k) \qquad \text{(chain rule)} \\
&= \underline{e}_i\,\frac{1}{2r}\,2\,x_i \\
&= \frac{1}{r}\,\underline{r} = \hat{\underline{r}}
\end{aligned}$$

The gradient of the length of the position vector is the unit vector pointing radially outwards from the origin. It is normal to the level surfaces which are spheres centered on the origin.

3.2.3 Identities for gradients

If $\phi(\underline{r})$ and $\psi(\underline{r})$ are real scalar fields, then:

1. **Distributive law**

$$\boxed{\underline{\nabla}\,\big(\phi(\underline{r})+\psi(\underline{r})\big) \;=\; \underline{\nabla}\,\phi(\underline{r})+\underline{\nabla}\,\psi(\underline{r})}$$

Proof:

$$\underline{\nabla}\,\big(\phi(\underline{r})\,+\,\psi(\underline{r})\big) \;=\; \underline{e}_i\,\frac{\partial}{\partial x_i}\,\big(\phi(\underline{r})\,+\,\psi(\underline{r})\big) \;=\; \underline{\nabla}\,\phi(\underline{r})\,+\,\underline{\nabla}\,\psi(\underline{r})$$

2. **Product rule**

$$\boxed{\underline{\nabla}\,\big(\phi(\underline{r})\;\psi(\underline{r})\big) \;=\; \psi(\underline{r})\;\underline{\nabla}\,\phi(\underline{r})+\phi(\underline{r})\;\underline{\nabla}\,\psi(\underline{r})}$$

Proof:

$$\begin{aligned}
\underline{\nabla}\,\big(\phi(\underline{r})\;\psi(\underline{r})\big) &= \underline{e}_i\,\frac{\partial}{\partial x_i}\,\big(\phi(\underline{r})\;\psi(\underline{r})\big)\\
&= \underline{e}_i\left(\psi(\underline{r})\,\frac{\partial \phi(\underline{r})}{\partial x_i}\;+\;\phi(\underline{r})\,\frac{\partial \psi(\underline{r})}{\partial x_i}\right)\\
&= \psi(\underline{r})\;\underline{\nabla}\,\phi(\underline{r})\;+\;\phi(\underline{r})\;\underline{\nabla}\,\psi(\underline{r})
\end{aligned}$$

3. **Chain rule:** If $F(\phi(\underline{r}))$ is a scalar field, then

$$\boxed{\underline{\nabla}\,F(\phi(\underline{r})) \;=\; \frac{\partial F(\phi)}{\partial \phi}\;\underline{\nabla}\,\phi(\underline{r})}$$

Proof:

$$\underline{\nabla}\,F(\phi(\underline{r})) \;=\; \underline{e}_i\,\frac{\partial}{\partial x_i}\;F(\phi(\underline{r})) \;=\; \underline{e}_i\,\frac{\partial F(\phi)}{\partial \phi}\;\frac{\partial \phi(\underline{r})}{\partial x_i} \;=\; \frac{\partial F(\phi)}{\partial \phi}\;\underline{\nabla}\,\phi(\underline{r})$$

Example of Chain Rule: If $\phi(\underline{r}) \;=\; r$ we can use result 3 from section 13.2 to give

$$\underline{\nabla}\,F(r) \;=\; \frac{\partial F(r)}{\partial r}\;\underline{\nabla}\,r \;=\; \frac{\partial F(r)}{\partial r}\;\frac{1}{r}\;\underline{r}$$

If $F(\phi(\underline{r})) = \phi(\underline{r})^n = r^n$ we find that

$$\underline{\nabla}\,(r^n) \;=\; (n\,r^{n-1})\,\frac{1}{r}\,\underline{r} \;=\; (n\,r^{n-2})\,\underline{r}$$

In particular

$$\underline{\nabla}\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$$

3.2.4 Transformation of the gradient

Here we prove the claim that the gradient actually is a vector (so far we assumed it was!).

Let the point P have coordinates x_i in the $\underline{e}_i$ basis and the *same* point P have coordinates x'_i in the $\underline{e}_i{}'$ basis i.e. we consider the vector transformation law $x_i \to x'_i = \lambda_{ij}\, x_j$.

$\phi(\underline{r})$ is a scalar if it depends only on the physical point P and not on the coordinates x_i or x'_i used to specify P. The *value* of ϕ at P is *invariant* under a change of basis λ (but the function may look different).

$$\phi(x_1,\, x_2,\, x_3) \to \phi'(x'_1,\, x'_2,\, x'_3) = \phi(x_1,\, x_2,\, x_3)$$

Now consider $\underline{\nabla}\phi$ in the new (primed) basis. Its components are

$$\frac{\partial}{\partial x'_i}\, \phi'(x'_1,\, x'_2,\, x'_3)$$

Using the chain rule, we obtain

$$\frac{\partial}{\partial x'_i}\, \phi'(x'_1,\, x'_2,\, x'_3) = \frac{\partial x_j}{\partial x'_i}\, \frac{\partial}{\partial x_j}\, \phi(x_1,\, x_2,\, x_3)\ .$$

Since $x_j = \lambda_{kj}\, x'_k$ (inverse vector transformation law)

$$\frac{\partial x_j}{\partial x'_i} = \lambda_{kj}\, \frac{\partial x'_k}{\partial x'_i} = \lambda_{kj}\, \delta_{ik} = \lambda_{ij}\ .$$

Hence

$$\frac{\partial}{\partial x'_i}\, \phi(x_1,\, x_2,\, x_3) = \lambda_{ij}\, \frac{\partial}{\partial x_j}\, \phi(x_1,\, x_2,\, x_3)\ .$$

which shows that the components of $\underline{\nabla}\phi$ respect the vector transformation law. Thus $\underline{\nabla}\phi(\underline{r})$ transforms as a **vector field** as claimed.

3.2.5 The Operator 'Del'

We can think of the **vector operator** $\underline{\nabla}$ (confusingly pronounced "del") acting on the **scalar field** $\phi(\underline{r})$ to produce the **vector field** $\underline{\nabla}\,\phi(\underline{r})$.

In Cartesians:

$$\boxed{\underline{\nabla} = \underline{e}_i\, \frac{\partial}{\partial x_i} = \underline{e}_1\, \frac{\partial}{\partial x_1} + \underline{e}_2\, \frac{\partial}{\partial x_2} + \underline{e}_3\, \frac{\partial}{\partial x_3}}$$

We call $\underline{\nabla}$ an 'operator' since it operates on something to its *right*. It is a vector operator since it has vector transformation properties. (More precisely it is a linear differential vector operator!)

We have seen how $\underline{\nabla}$ acts on a scalar field to produce a vector field. We can make products of the vector operator $\underline{\nabla}$ with other vector quantities to produce new operators and fields in the same way as we could make scalar and vector products of two vectors.

For example, recall that the directional derivative of ϕ in direction $\hat{\underline{s}}$ was given by $\hat{\underline{s}} \cdot \underline{\nabla}\phi$. Generally, we can interpret $\underline{A} \cdot \underline{\nabla}$ as a **scalar operator**:

$$\underline{A} \cdot \underline{\nabla} = A_i \frac{\partial}{\partial x_i}$$

i.e. $\underline{A} \cdot \underline{\nabla}$ acts on a scalar field to its *right* to produce another scalar field

$$(\underline{A} \cdot \underline{\nabla})\, \phi(\underline{r}) \;=\; A_i \frac{\partial \phi(\underline{r})}{\partial x_i} \;=\; A_1 \frac{\partial \phi(\underline{r})}{\partial x_1} + A_2 \frac{\partial \phi(\underline{r})}{\partial x_2} + A_3 \frac{\partial \phi(\underline{r})}{\partial x_3}$$

Actually we can also act with this operator on a vector field to get another vector field.

$$\begin{aligned} (\underline{A} \cdot \underline{\nabla})\, \underline{V}(\underline{r}) &= A_i \frac{\partial}{\partial x_i} \underline{V}(\underline{r}) \;=\; A_i \frac{\partial}{\partial x_i} \Big(V_j(\underline{r})\, \underline{e}_j \Big) \\ &= \underline{e}_1\, (\underline{A} \cdot \underline{\nabla})\, V_1(\underline{r}) + \underline{e}_2\, (\underline{A} \cdot \underline{\nabla})\, V_2(\underline{r}) + \underline{e}_3\, (\underline{A} \cdot \underline{\nabla})\, V_3(\underline{r}) \end{aligned}$$

The alternative expression $\underline{A} \cdot \big(\underline{\nabla}\, \underline{V}(\underline{r}) \big)$ is *undefined* because $\underline{\nabla}\, \underline{V}(\underline{r})$ doesn't make sense.

N.B. Great care is required with the order in products since, in general, products involving operators are not commutative. For example

$$\underline{\nabla} \cdot \underline{A} \;\neq\; \underline{A} \cdot \underline{\nabla}$$

$\underline{A} \cdot \underline{\nabla}$ is a scalar differential operator whereas $\underline{\nabla} \cdot \underline{A} = \dfrac{\partial A_i}{\partial x_i}$ gives a scalar field called the **divergence** of $\underline{A}$.

end of lecture 13

3.3 More on Vector Operators

In this lecture we combine the vector operator $\underline{\nabla}$ ('del') with a vector field to define two new operations 'div' and 'curl'. Then we define the Laplacian.

3.3.1 Divergence

We define the **divergence** of a vector field $\underline{A}$ (pronounced 'div A') as:-

$$\boxed{\text{div}\,\underline{A}(\underline{r}) \;\equiv\; \underline{\nabla}\cdot\underline{A}(\underline{r})}$$

In Cartesian coordinates

$$\boxed{\begin{aligned} \underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{\partial}{\partial x_i}\,A_i(\underline{r}) \quad &= \quad \frac{\partial A_1(\underline{r})}{\partial x_1} + \frac{\partial A_2(\underline{r})}{\partial x_2} + \frac{\partial A_3(\underline{r})}{\partial x_3} \\ \text{or} \quad &\quad \frac{\partial A_x(\underline{r})}{\partial x} + \frac{\partial A_y(\underline{r})}{\partial y} + \frac{\partial A_z(\underline{r})}{\partial z} \quad \text{in } x, y, z \text{ notation} \end{aligned}}$$

It is easy to show that $\underline{\nabla}\cdot\underline{A}(\underline{r})$ is a scalar field: Under a change of basis $\underline{e}_i \to \underline{e}_i{}' = \lambda_{ij}\,\underline{e}_j$

$$\begin{aligned} \big(\underline{\nabla}\cdot\underline{A}(\underline{r})\big)' \quad &= \quad \frac{\partial}{\partial x_i'}\,A_i'(x_1',\,x_2',\,x_3') \quad = \quad \frac{\partial x_j}{\partial x_i'}\,\frac{\partial}{\partial x_j}\,\big(\lambda_{ik}A_k(x_1,\,x_2,\,x_3)\big) \\ &= \quad \lambda_{ij}\,\lambda_{ik}\,\frac{\partial}{\partial x_j}\,A_k(x_1,\,x_2,\,x_3) \quad = \quad \delta_{jk}\,\frac{\partial}{\partial x_j}\,A_k(x_1,\,x_2,\,x_3) \\ &= \quad \frac{\partial}{\partial x_j}\,A_j(x_1,\,x_2,\,x_3) = \underline{\nabla}\cdot\underline{A}(\underline{r}) \end{aligned}$$

Hence $\underline{\nabla}\cdot\underline{A}$ is invariant under a change of basis and is thus a **scalar field**.

Example: $\underline{A}(\underline{r}) = \underline{r} \quad \Rightarrow \quad \boxed{\underline{\nabla}\cdot\underline{r} \;=\; 3}$ a *very* useful & important result!

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 1 + 1 + 1 = 3$$

In suffix notation

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

Example: Here we use 'xyz' notation: $x_1 = x,\ x_2 = y,\ x_3 = z$. Consider $\underline{A} = x^2 z\,\underline{e}_1 - 2y^3z^2\,\underline{e}_2 + xy^2z\,\underline{e}_3$

$$\begin{aligned} \underline{\nabla}\cdot\underline{A} \quad &= \quad \frac{\partial}{\partial x}\,(x^2z) \;-\; \frac{\partial}{\partial y}\,(2y^3z^2) \;+\; \frac{\partial}{\partial z}\,(xy^2z) \\ &= \quad 2xz - 6y^2z^2 + xy^2 \end{aligned}$$

Thus for instance at the point (1,1,1) $\underline{\nabla}\cdot\underline{A} = 2 - 6 + 1 = -3$.

3.3.2 Curl

We define the curl of a vector field, curl $\underline{A}$, as

$$\boxed{\text{curl}\,\underline{A}(\underline{r}) \;\equiv\; \underline{\nabla} \times \underline{A}(\underline{r})}$$

Note that curl $\underline{A}$ is a *vector* field

In Cartesian coordinates

$$\begin{aligned} \underline{\nabla} \times \underline{A} &= \underline{e}_i \, \left(\underline{\nabla} \times \underline{A}\right)_i \\ &= \underline{e}_i \, \epsilon_{ijk} \, \frac{\partial}{\partial x_j} \, A_k \end{aligned}$$

ie, the ith component of $\underline{\nabla} \times \underline{A}$ is

$$\boxed{\left(\underline{\nabla} \times \underline{A}\right)_i \;=\; \epsilon_{ijk} \, \frac{\partial}{\partial x_j} \, A_k}$$

More explicitly

$$\left(\underline{\nabla} \times \underline{A}\right)_1 \;=\; \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \quad \text{etc,}$$

Instead of the above equation for curl that uses ϵ_{ijk}, one can use a determinant form (c.f. the expression of the vector product)

$$\boxed{\underline{\nabla} \times \underline{A} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \,.}$$

Example: $\underline{A}(\underline{r}) = \underline{r} \quad \Rightarrow \quad \boxed{\underline{\nabla} \times \underline{r} \;=\; 0}$ another *very* useful & important result!

$$\begin{aligned} \underline{\nabla} \times \underline{r} &= \underline{e}_i \, \epsilon_{ijk} \, \frac{\partial}{\partial x_j} \, x_k \\ &= \underline{e}_i \, \epsilon_{ijk} \, \delta_{jk} = \underline{e}_i \, \epsilon_{ijj} = 0 \end{aligned}$$

or, using the determinant formula, $\underline{\nabla} \times \underline{r} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ x_1 & x_2 & x_3 \end{vmatrix} \;\equiv\; 0$

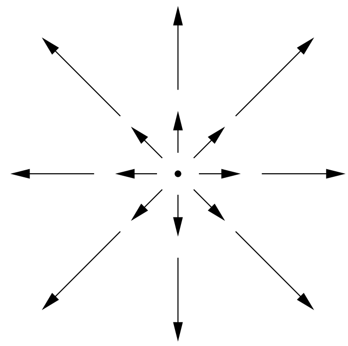
Example: Compute the curl of $\underline{V} = x^2 y \underline{e}_1 + y^2 x \underline{e}_2 + xyz \underline{e}_3$:

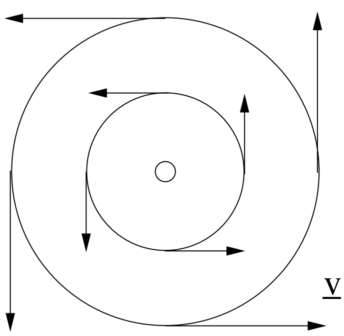
$$\underline{\nabla} \times \underline{V} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^2 x & xyz \end{vmatrix} = \underline{e}_1 (xz - 0) - \underline{e}_2 (yz - 0) + \underline{e}_3 (y^2 - x^2)$$

3.3.3 Physical Interpretation of 'div' and 'curl'

Full interpretations of the divergence and curl of a vector field are best left until after we have studied the Divergence Theorem and Stokes' Theorem respectively. However, we can gain some intuitive understanding by looking at simple examples where div and/or curl vanish.

First consider the radial field $\underline{A} = \underline{r}$; $\underline{\nabla} \cdot \underline{A} = 3$; $\underline{\nabla} \times \underline{A} = 0$. We sketch the vector field $\underline{A}(\underline{r})$ by drawing at selected points vectors of the appropriate direction and magnitude. These give the tangents of 'flow lines'. Roughly speaking, in this example the divergence is positive because bigger arrows come out of a point than go in. So the field 'diverges'. (Once the concept of flux of a vector field is understood this will make more sense.)





Now consider the field $\underline{v} = \underline{\omega} \times \underline{r}$ where $\underline{\omega}$ is a constant vector. One can think of $\underline{v}$ as the velocity of a point in a rigid rotating body. We sketch a cross-section of the field $\underline{v}$ with $\underline{\omega}$ chosen to point out of the page. We can calculate $\underline{\nabla} \times \underline{v}$ as follows:

$$\begin{aligned}
\underline{\nabla} \times \left(\underline{\omega} \times \underline{r}\right) &= \underline{e}_i\, \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\underline{\omega} \times \underline{r}\right)_k \quad = \underline{e}_i\, \epsilon_{ijk} \frac{\partial}{\partial x_j}\, \epsilon_{klm}\, \omega_l\, x_m \\
&= \underline{e}_i \left(\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}\right) \omega_l\, \delta_{jm} \qquad \left(\text{since } \frac{\partial \omega_l}{\partial x_j} = 0\right) \\
&= \underline{e}_i \left(\omega_i\, \delta_{jj} - \delta_{ij}\, \omega_j\right) = \underline{e}_i\, 2\, \omega_i = 2\underline{\omega}
\end{aligned}$$

Thus we obtain yet another *very* useful & important result:

$$\boxed{\underline{\nabla} \times \left(\underline{\omega} \times \underline{r}\right) = 2\underline{\omega}}$$

To understand intuitively the non-zero curl imagine that the flow lines are those of a rotating fluid with a small ball centred on a flow line of the field. The centre of the ball will follow the flow line. However the effect of the neighbouring flow lines is to make the ball rotate. Therefore the field has non-zero 'curl' and the axis of rotation gives the direction of the curl. In the previous example ($\underline{A} = \underline{r}$) the ball would just move away from origin without rotating therefore the field $\underline{r}$ has zero curl.

Terminology:

1. If $\underline{\nabla} \cdot \underline{A}(\underline{r}) = 0$ in some region R, $\underline{A}$ is said to be **solenoidal** in R.

2. If $\underline{\nabla} \times \underline{A}(\underline{r}) = 0$ in some region R, $\underline{A}$ is said to be **irrotational** in R.

3.3.4 The Laplacian Operator ∇^2

We may take the *divergence* of the *gradient* of a scalar field $\phi(\underline{r})$

$$\underline{\nabla} \cdot (\underline{\nabla}\, \phi(\underline{r})) \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}\, \phi(\underline{r}) \;\equiv\; \nabla^2 \phi(\underline{r})$$

∇^2 is the **Laplacian operator**, pronounced 'del-squared'. In Cartesian coordinates

$$\nabla^2 \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}$$

More explicitly

$$\boxed{\nabla^2\, \phi(\underline{r}) \;=\; \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}}$$

It may be shown that the Laplacian of a scalar field $\nabla^2\, \phi$ is also a scalar field, *i.e.* the Laplacian is a **scalar operator**.

Example

$$\nabla^2\, r^2 = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} x_j x_j = \frac{\partial}{\partial x_i}(2x_i) = 2\delta_{ii} = 6\ .$$

In Cartesian coordinates, the effect of the Laplacian on a vector field $\underline{A}$ is *defined* to be

$$\nabla^2 \underline{A}(\underline{r}) \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}\, \underline{A}(\underline{r}) \;=\; \frac{\partial^2}{\partial x_1^2}\, \underline{A}(\underline{r}) \;+\; \frac{\partial^2}{\partial x_2^2}\, \underline{A}(\underline{r}) \;+\; \frac{\partial^2}{\partial x_3^2}\, \underline{A}(\underline{r})$$

The Laplacian acts on a vector field to produce another vector field.

end of lecture 14

3.4 Vector Operator Identities

There are many identities involving div, grad, and curl. It is not necessary to know *all* of these, but you are advised to be able to produce from memory expressions for $\underline{\nabla} r$, $\underline{\nabla} \cdot \underline{r}$, $\underline{\nabla} \times \underline{r}$, $\underline{\nabla}\phi(r)$, $\underline{\nabla}(\underline{a} \cdot \underline{r})$, $\underline{\nabla} \times (\underline{a} \times \underline{r})$, $\underline{\nabla}(fg)$, and first four identities given below. You should be *familiar* with the rest and to be able to *derive* and *use* them when necessary!

Most importantly you should be at ease with div, grad and curl. This only comes through practice and deriving the various identities gives you just that. In these derivations the advantages of suffix notation, the summation convention and ϵ_{ijk} will become apparent.

In what follows, $\phi(\underline{r})$ is a scalar field; $\underline{A}(\underline{r})$ and $\underline{B}(\underline{r})$ are vector fields.

3.4.1 Distributive Laws

1. $\underline{\nabla} \cdot (\underline{A} + \underline{B}) \ = \ \underline{\nabla} \cdot \underline{A} + \underline{\nabla} \cdot \underline{B}$
2. $\underline{\nabla} \times (\underline{A} + \underline{B}) \ = \ \underline{\nabla} \times \underline{A} + \underline{\nabla} \times \underline{B}$

The proofs of these are straightforward using suffix or 'x y z' notation and follow from the fact that div and curl are linear operations.

3.4.2 Product Laws

The results of taking the div or curl of **products** of vector and scalar fields are predictable but need a little care:-

3. $\underline{\nabla} \cdot (\phi\, \underline{A}) \ = \ \phi\, \underline{\nabla} \cdot \underline{A} + \underline{A} \cdot \underline{\nabla}\phi$
4. $\underline{\nabla} \times (\phi\, \underline{A}) \ = \ \phi\, (\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A} \ = \ \phi\, (\underline{\nabla} \times \underline{A}) - \underline{A} \times \underline{\nabla}\phi$

Proof of (4): first using ϵ_{ijk}

$$\begin{aligned}
\underline{\nabla} \times (\phi\, \underline{A}) &= \underline{e}_i\, \epsilon_{ijk} \frac{\partial}{\partial x_j} (\phi\, A_k) \\
&= \underline{e}_i\, \epsilon_{ijk} \left(\phi \left(\frac{\partial A_k}{\partial x_j} \right) + \left(\frac{\partial \phi}{\partial x_j} \right) A_k \right) \\
&= \phi\, (\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A}
\end{aligned}$$

or avoiding ϵ_{ijk} and using 'x y z' notation: $\underline{\nabla} \times (\phi\, \underline{A}) = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix}$.

The x component is given by

$$\begin{aligned} \frac{\partial(\phi A_z)}{\partial y} - \frac{\partial(\phi A_y)}{\partial z} &= \phi(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \left(\frac{\partial \phi}{\partial y}\right) A_z - \left(\frac{\partial \phi}{\partial z}\right) A_y \\ &= \phi(\underline{\nabla} \times \underline{A})_x + \big[(\underline{\nabla}\phi) \times \underline{A}\big]_x \end{aligned}$$

A similar proof holds for the y and z components.

Although we have used Cartesian coordinates in our proofs, the identities hold in all coordinate systems.

3.4.3 Products of Two Vector Fields

Things start getting complicated!

5. $\underline{\nabla}\,(\underline{A} \cdot \underline{B}) \;=\; (\underline{A} \cdot \underline{\nabla})\,\underline{B} \;+\; (\underline{B} \cdot \underline{\nabla})\underline{A} \;+\; \underline{A} \times (\underline{\nabla} \times \underline{B}) \;+\; \underline{B} \times (\underline{\nabla} \times \underline{A})$
6. $\underline{\nabla} \cdot (\underline{A} \times \underline{B}) \;=\; \underline{B} \cdot (\underline{\nabla} \times \underline{A}) \;-\; \underline{A} \cdot (\underline{\nabla} \times \underline{B})$
7. $\underline{\nabla} \times (\underline{A} \times \underline{B}) \;=\; \underline{A}\,(\underline{\nabla} \cdot \underline{B}) \;-\; \underline{B}\,(\underline{\nabla} \cdot \underline{A}) \;+\; (\underline{B} \cdot \underline{\nabla})\,\underline{A} \;-\; (\underline{A} \cdot \underline{\nabla})\,\underline{B}$

Proof of (6):

$$\begin{aligned} \underline{\nabla} \cdot (\underline{A} \times \underline{B}) &= \frac{\partial}{\partial x_i}\,\epsilon_{ijk}\, A_j\, B_k \\ &= \epsilon_{ijk} \left(\frac{\partial A_j}{\partial x_i}\right) B_k \; + \epsilon_{ijk}\, A_j \left(\frac{\partial B_k}{\partial x_i}\right) \\ &= B_k \epsilon_{kij}\, \frac{\partial A_j}{\partial x_i} \; - A_j \epsilon_{jik}\, \frac{\partial B_k}{\partial x_i} \end{aligned}$$

The proofs of (5) and (7) involve the product of two epsilon symbols. For example, this is why there are four terms on the rhs of (7).

All other results involving one $\underline{\nabla}$ can be derived from the above identities.

Example: If $\underline{a}$ is a *constant* vector, and $\underline{r}$ is the position vector, show that

$$\underline{\nabla}\,(\underline{a} \cdot \underline{r}) = (\underline{a} \cdot \underline{\nabla})\,\underline{r} = \underline{a}$$

In lecture 13 we showed that $\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = \underline{a}$ for constant $\underline{a}$. Hence, we need only evaluate

$$(\underline{a}\cdot\underline{\nabla})\,\underline{r} \;=\; a_i\,\frac{\partial}{\partial x_i}\,\underline{e}_j\,x_j \;=\; a_i\,\underline{e}_j\,\delta_{ij} \;=\; a_i\,\underline{e}_i \;=\; \underline{a} \qquad (31)$$

and the identity holds.

Example: Show that $\underline{\nabla}\cdot(\underline{\omega}\times\underline{r}) = 0$ where $\underline{\omega}$ is a *constant* vector.
Using (6) $\underline{\nabla}\cdot(\underline{\omega}\times\underline{r}) = \underline{r}\cdot(\underline{\nabla}\times\underline{\omega}) \;-\; \underline{\omega}\cdot(\underline{\nabla}\times\underline{r}) = 0 - 0$

Example: Show that $\underline{\nabla}\cdot(r^{-3}\underline{r}) = 0$, for $r \neq 0$ (where $r = |\underline{r}|$ as usual).
Using identity (3), we have

$$\underline{\nabla}\cdot(r^{-3}\underline{r}) \;=\; r^{-3}\,(\underline{\nabla}\cdot\underline{r}) \;+\; \underline{r}\cdot\underline{\nabla}(r^{-3})$$

We have previously shown that $\underline{\nabla}\cdot\underline{r} = 3$ and that $\underline{\nabla}(r^n) = n\,r^{n-2}\,\underline{r}$. Hence

$$\begin{aligned}
\underline{\nabla}\cdot(r^{-3}\underline{r}) &= r^{-3}\,(\underline{\nabla}\cdot\underline{r}) \;+\; \underline{r}\cdot\underline{\nabla}(r^{-3}) \\
&= \frac{3}{r^3} + \underline{r}\cdot\left(\frac{-3}{r^5}\,\underline{r}\right) \\
&= \frac{3}{r^3} + \frac{-3}{r^5}\,r^2 \;=\; 0 \qquad (\text{except at}\, r = 0)
\end{aligned}$$

3.4.4 Identities involving 2 gradients

8. $\underline{\nabla}\times(\underline{\nabla}\phi) \;=\; 0$ — curl grad ϕ is always zero.
9. $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{A}) \;=\; 0$ — div curl $\underline{A}$ is always zero.
10. $\underline{\nabla}\times(\underline{\nabla}\times\underline{A}) \;=\; \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) \;-\; \nabla^2\underline{A}$

Proofs are easily obtained in Cartesian coordinates using suffix notation:-

Proof of (8)

$$\begin{aligned}
\underline{\nabla}\times(\underline{\nabla}\,\phi) &= \underline{e}_i\,\epsilon_{ijk}\,\frac{\partial}{\partial x_j}\,(\underline{\nabla}\,\phi)_k \;=\; \underline{e}_i\,\epsilon_{ijk}\,\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k}\,\phi \\
&= \underline{e}_i\,\epsilon_{ijk}\,\frac{\partial}{\partial x_k}\,\frac{\partial}{\partial x_j}\,\phi \qquad \left(\text{since}\;\frac{\partial^2\phi}{\partial x\partial y} \;=\; \frac{\partial^2\phi}{\partial y\partial x}\;\text{etc}\right) \\
&= \underline{e}_i\,\epsilon_{ikj}\,\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k}\,\phi \qquad (\text{interchanging labels}\; j \;\text{and}\; k) \\
&= -\underline{e}_i\,\epsilon_{ijk}\,\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k}\,\phi \qquad (ikj \to ijk \;\text{gives minus sign}) \\
&= -\underline{\nabla}\times(\underline{\nabla}\,\phi) \;=\; 0
\end{aligned}$$

since any vector equal to minus itself is must be zero. The proof of (9) is similar. It is important to understand how these two identities stem from the anti-symmetry of ϵ_{ijk}.

Proof of (10)

$$
\begin{aligned}
\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) &= \underline{e}_i \, \epsilon_{ijk} \, \frac{\partial}{\partial x_j} \, (\underline{\nabla} \times \underline{A})_k \\
&= \underline{e}_i \, \epsilon_{ijk} \, \frac{\partial}{\partial x_j} \, \epsilon_{klm} \, \frac{\partial}{\partial x_l} \, A_m \\
&= \underline{e}_i \, \left(\left(\delta_{il} \, \delta_{jm} - \delta_{im} \, \delta_{jl} \right) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} \, A_m \right) \\
&= \underline{e}_i \, \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \, A_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \, A_i \right) \\
&= \underline{e}_i \, \left(\frac{\partial}{\partial x_i} \, (\underline{\nabla} \cdot \underline{A}) - \nabla^2 A_i \right) \\
&= \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \, \underline{A}
\end{aligned}
$$

Although this proof look tedious it is far simpler than trying to use 'xyz' (try both and see!). It is an important result and is used frequently in electromagnetism, fluid mechanics, and other 'field theories'.

Finally, when a scalar field ϕ depends only on the magnitude of the position vector $r = |\underline{r}|$, we have

$$
\nabla^2 \, \phi(r) = \phi''(r) + \frac{2\phi'(r)}{r}
$$

where the prime denotes differentiation with respect to r. Proof of this relation is left to the tutorial.

3.4.5 Polar Co-ordinate Systems

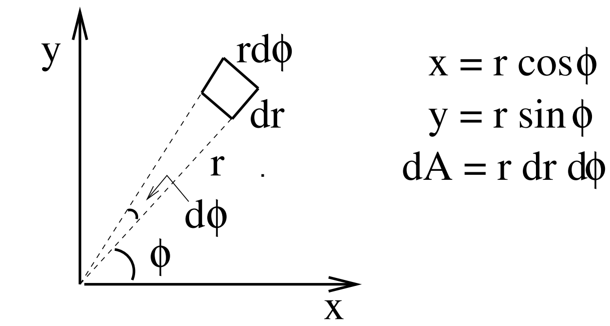
Before commencing with integral vector calculus we review here polar co-ordinate systems. Here dV indicates a volume element and dA an area element. Note that different conventions, *e.g.* for the angles ϕ and θ, are sometimes used, in particular in the Mathematics 'Several Variable Calculus' Module.

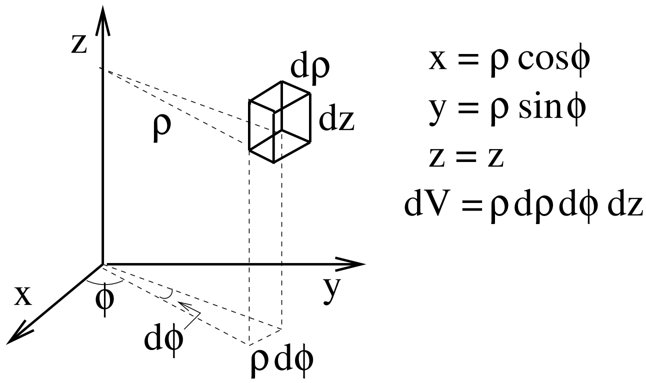
Plane polar co-ordinates

Cylindrical polar co-ordinates

Spherical polar co-ordinates

end of lecture 15





4 Integrals over Fields

4.1 Scalar and Vector Integration and Line Integrals

4.1.1 Scalar & Vector Integration

You should already be familar with integration in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$. Here we review integration of a scalar field with an example.

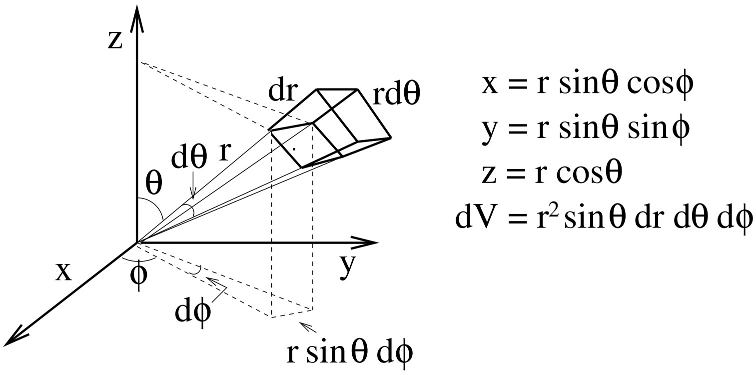
Consider a hemisphere of radius a centered on the $\underline{e}_3$ axis and with bottom face at $z = 0$. If the mass density (a scalar field) is $\rho(r) = \sigma/r$ where σ is a constant, then what is the total mass?

It is most convenient to use spherical polars (see lecture 15). Then

$$M = \int_{\text{hemisphere}} \rho(\underline{r})dV = \int_0^a r^2\rho(r)dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi\sigma \int_0^a rdr = \pi\sigma a^2$$

Now consider the centre of mass vector

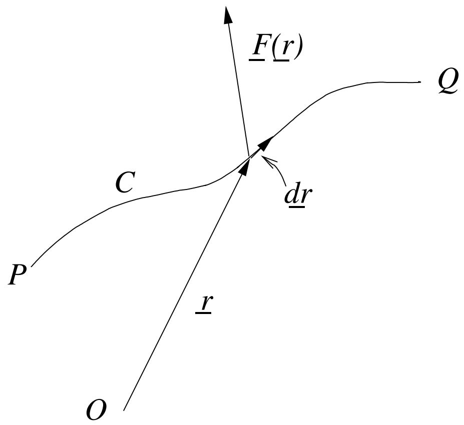
$$M\underline{R} = \int_V \underline{r}\rho(\underline{r})dV$$



This is our first example of integrating a vector field (here $\underline{r}\rho(\underline{r})$). To do so simply integrate each component using $\underline{r} = r\sin\theta\cos\phi\underline{e}_1 + r\sin\theta\sin\phi\underline{e}_2 + r\ \cos\theta\underline{e}_3$

$$\begin{aligned}
MX &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \cos\phi\ d\phi = 0 \quad \text{since } \phi \text{ integral gives } 0 \\
MY &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi\ d\phi = 0 \quad \text{since } \phi \text{ integral gives } 0 \\
MZ &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = 2\pi\sigma \int_0^a r^2 dr \int_0^{\pi/2} \frac{\sin 2\theta}{2} d\theta \\
&= \frac{2\pi\sigma a^3}{3}\left[\frac{-\cos 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi\sigma a^3}{3} \qquad \therefore \quad \underline{R} = \frac{a}{3}\underline{e}_3
\end{aligned}$$

4.1.2 Line Integrals



As an example, consider a particle constrained to move on a wire say. Only the component of the force along the wire does any work. Therefore the work done in moving the particle from $\underline{r}$ to $\underline{r} + d\underline{r}$ is

$$dW = \underline{F} \cdot d\underline{r}\ .$$

The total work done in moving particle along a wire which follows some curve C between two points P, Q is

$$W_C = \int_P^Q dW = \int_C \underline{F}(\underline{r}) \cdot d\underline{r}\ .$$

This is a line integral along the curve C.

More generally let $\underline{A}(\underline{r})$ be a vector field defined in the region R, and let C be a curve in R joining two points P and Q. $\underline{r}$ is the position vector at some point on the curve; $d\underline{r}$ is an infinitesimal vector *along* the curve at $\underline{r}$.

The magnitude of $d\underline{r}$ is the infinitesimal **arc length**: $ds = \sqrt{d\underline{r} \cdot d\underline{r}}$.

We define $\underline{\hat{t}}$ to be the **unit vector** tangent to the curve at $\underline{r}$ (points in the direction of $\underline{dr}$)

$$\underline{\hat{t}} \;=\; \frac{d\underline{r}}{ds}$$

Note that, in general, $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ **depends on the path** joining P and Q.

In Cartesian coordinates, we have

$$\int_C \underline{A}\cdot\underline{dr} \;=\; \int_C A_i dx_i \;=\; \int_C (A_1 dx_1 \,+\, A_2 dx_2 \,+\, A_3 dx_3)$$

4.1.3 Parametric Representation of a line integral

Often a curve in 3d can be parameterised by a single parameter e.g. if the curve were the trajectory of a particle then time would be the parameter. Sometimes the parameter of a line integral is chosen to be the arc-length s along the curve C.

Generally for parameterisation by λ (varying from λ_P to λ_Q)

$$x_i \;=\; x_i(\lambda), \quad \text{with } \lambda_P \le \lambda \le \lambda_Q$$

then

$$\int_C \underline{A}\cdot\underline{dr} \;=\; \int_{\lambda_P}^{\lambda_Q}\left(\underline{A}\cdot\frac{d\underline{r}}{d\lambda}\right)d\lambda = \int_{\lambda_P}^{\lambda_Q}\left(A_1\,\frac{dx_1}{d\lambda} \,+\, A_2\,\frac{dx_2}{d\lambda} \,+\, A_3\,\frac{dx_3}{d\lambda}\right)d\lambda$$

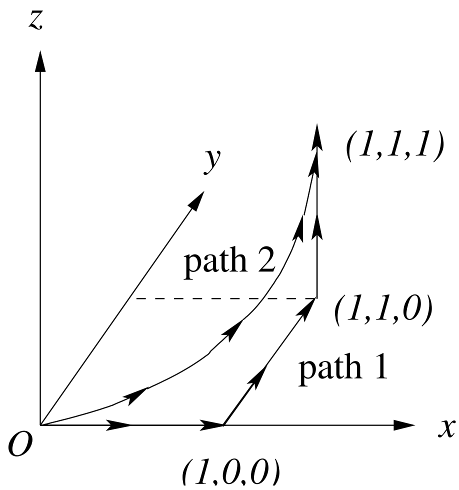
If necessary, the curve C may be subdivided into sections, each with a different parameterisation (piecewise smooth curve).

Example: $\underline{A} = (3x^2+6y)\,\underline{e}_1 - 14yz\underline{e}_2 + 20xz^2\underline{e}_3$. Evaluate $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ between the points with Cartesian coordinates $(0,0,0)$ and $(1,1,1)$, along the paths C:

1. $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$ (straight lines).
2. $x=\lambda$, $y=\lambda^2$ $z=\lambda^3$; from $\lambda=0$ to $\lambda=1$.

1.
   - Along the line from $(0,0,0)$ to $(1,0,0)$, we have $y=z=0$, so $dy=dz=0$, hence $\underline{dr} = \underline{e}_1\,dx$ and $\underline{A} = 3x^2\,\underline{e}_1$, (here the parameter is x):

$$\int_{(0,0,0)}^{(1,0,0)} \underline{A}\cdot\underline{dr} \;=\; \int_{x=0}^{x=1} 3x^2\,dx \;=\; \left[x^3\right]_0^1 \;=\; 1$$



- Along the line from $(1,0,0)$ to $(1,1,0)$, we have $x = 1$, $dx = 0$, $z = dz = 0$, so $\underline{dr} = \underline{e}_2\, dy$ (here the parameter is y) and

$$\underline{A} \;=\; \left(3x^2 + 6y\right)\Big|_{x=1}\, \underline{e}_1 \;=\; (3+6y)\,\underline{e}_1.$$

$$\int_{(1,0,0)}^{(1,1,0)} \underline{A}\cdot\underline{dr} \;=\; \int_{y=0}^{y=1} (3+6y)\; \underline{e}_1\cdot\underline{e}_2\, dy \;=\; 0.$$

- Along the line from $(1,1,0)$ to $(1,1,1)$, we have $x = y = 1$, $dx = dy = 0$, and hence $\underline{dr} = \underline{e}_3\, dz$ and $\underline{A} = 9\,\underline{e}_1 - 14z\,\underline{e}_2 + 20z^2\,\underline{e}_3$, therefore

$$\int_{(1,1,0)}^{(1,1,1)} \underline{A}\cdot\underline{dr} \;=\; \int_{z=0}^{z=1} 20z^2\, dz \;=\; \left[\frac{20}{3}z^3\right]_0^1 \;=\; \frac{20}{3}$$

Adding up the 3 contributions we get

$$\int_C \underline{A}\cdot\underline{dr} \;=\; 1 \,+\, 0 \,+\, \frac{20}{3} \;=\; \frac{23}{3} \qquad \text{along path (1)}$$

2. To integrate $\underline{A} = (3x^2+6y)\,\underline{e}_1 - 14yz\underline{e}_2 + 20xz^2\underline{e}_3$ along path (2) (where the parameter is λ), we write

$$\begin{aligned}
\underline{r} &= \lambda\,\underline{e}_1 + \lambda^2\,\underline{e}_2 + \lambda^3\,\underline{e}_3 \\
\frac{\underline{dr}}{d\lambda} &= \underline{e}_1 \,+\, 2\lambda\,\underline{e}_2 \,+\, 3\lambda^2\,\underline{e}_3 \\
\underline{A} &= \left(3\lambda^2 + 6\lambda^2\right)\,\underline{e}_1 \,-\, 14\lambda^5\,\underline{e}_2 \,+\, 20\lambda^7\,\underline{e}_3 \quad \text{so that}
\end{aligned}$$

$$\int_C \left(\underline{A}\cdot\frac{\underline{dr}}{d\lambda}\right)\, d\lambda = \int_{\lambda=0}^{\lambda=1} \left(9\lambda^2 \,-\, 28\lambda^6 \,+\, 60\lambda^9\right)\, d\lambda = \left[3\lambda^3 \,-\, 4\lambda^7 + 6\lambda^{10}\right]_0^1 \;=\; 5$$

$$\text{Hence} \qquad \int_C \underline{A}\cdot\underline{dr} \;=\; 5 \qquad \text{along path (2)}$$

In this case, the integral of $\underline{A}$ from $(0,0,0)$ to $(1,1,1)$ depends on the path taken.

The line integral $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ is a **scalar** quantity. Another **scalar** line integral is $\displaystyle\int_C f\,ds$ where $f(\underline{r})$ is a scalar field and ds is the infinitesimal arc-length introduced earlier.

Line integrals around a **simple** (doesn't intersect itself) **closed** curve C are denoted by $\displaystyle\oint_C$

$$\text{e.g.} \qquad \oint_C \underline{A}\cdot\underline{dr} \qquad \equiv \text{ the \textbf{circulation} of } \underline{A} \text{ around } C$$

Example : Let $f(\underline{r}) = ax^2 + by^2$. Evaluate $\displaystyle\oint_C f\,ds$ around the unit circle C in the $x-y$ plane, centred on the origin:

$$x = \cos\phi,\ y = \sin\phi,\ z = 0; \quad 0 \le \phi \le 2\pi.$$

$$\begin{aligned}
\text{We have} \qquad f(\underline{r}) &= ax^2 + by^2 \;=\; a\cos^2\phi \,+\, b\sin^2\phi \\
\underline{r} &= \cos\phi\,\underline{e}_1 \,+\, \sin\phi\,\underline{e}_2 \\
\underline{dr} &= (-\sin\phi\,\underline{e}_1 \,+\, \cos\phi\,\underline{e}_2)\ d\phi \\
\text{so} \quad ds &= \sqrt{\underline{dr}\cdot\underline{dr}} \;=\; \left(\cos^2\phi \,+\, \sin^2\phi\right)^{1/2} d\phi \;=\; d\phi
\end{aligned}$$

Therefore, for this example,

$$\oint_C f\,ds \;=\; \int_0^{2\pi} \left(a\cos^2\phi \,+\, b\sin^2\phi\right) d\phi \;=\; \pi\,(a\,+\,b)$$

The **length** s of a curve C is given by $s = \displaystyle\int_C ds$. In this example $s = 2\pi$.

We can also define **vector** line integrals e.g.:-

1. $\displaystyle\int_C \underline{A}\,ds \;=\; \underline{e}_i \int_C A_i\,ds$ in Cartesian coordinates.

2. $\displaystyle\int_C \underline{A}\times\underline{dr} \;=\; \underline{e}_i\,\epsilon_{ijk}\int_C A_j\,dx_k$ in Cartesians.

Example : Consider a current of magnitude I flowing along a wire following a closed path C. The magnetic force on an element $\underline{dr}$ of the wire is $I\underline{dr}\times\underline{B}$ where $\underline{B}$ is the magnetic field at $\underline{r}$. Let $\underline{B}(\underline{r}) = x\,\underline{e}_1 + y\,\underline{e}_2$. Evaluate $\oint_C \underline{B} \times \underline{dr}$ for a circular current loop of radius a in the $x - y$ plane, centred on the origin.

$$\begin{aligned}\underline{B} &= a\cos\phi\,\underline{e}_1 \;+\; a\sin\phi\,\underline{e}_2 \\ \underline{dr} &= (-a\sin\phi\,\underline{e}_1 \;+\; a\cos\phi\,\underline{e}_2)\; d\phi\end{aligned}$$

$$\text{Hence} \qquad \oint_C \underline{B} \times \underline{dr} \;=\; \int_0^{2\pi} \left(a^2\cos^2\phi \,+\, a^2\sin^2\phi\right)\,\underline{e}_3\,d\phi \;=\; \underline{e}_3\,a^2\int_0^{2\pi} d\phi \;=\; 2\pi a^2\,\underline{e}_3$$

end of lecture 16

4.2 The Scalar Potential

Consider again the work done by a force. If the force is *conservative*, i.e. total energy is conserved, then the work done is equal to minus the change in potential energy

$$dV = -dW = -\underline{F}\cdot\underline{dr} = -F_i dx_i$$

Now we can also write dV as

$$dV = \frac{\partial V}{\partial x_i} dx_i = (\underline{\nabla} V)_i dx_i$$

Therefore we can identify

$$\boxed{\underline{F} = -\underline{\nabla} V}$$

Thus the force is minus the gradient of the (scalar) potential. The minus sign is conventional and chosen so that potential energy decreases as the force does work.

In this example we knew that a potential existed (we postulated conservation of energy). More generally we would like to know under what conditions can a vector field $\underline{A}(\underline{r})$ be written as the gradient of a scalar field ϕ, *i.e.* when does $\underline{A}(\underline{r}) \;=\; (\pm)\,\underline{\nabla}\,\phi(\underline{r})$ hold?

Aside: A **simply connected** region R is a region where every closed curve in R can be shrunk continuously to a point while remaining entirely in R. The inside of a sphere is simply connected while the region between two concentric cylinders is **not** simply connected: it is doubly connected. For this course we shall be concerned with simply connected regions

4.2.1 Theorems on Scalar Potentials

For a vector field $\underline{A}(\underline{r})$ defined in a simply connected region R, the following three statements are equivalent, *i.e.*, **any one implies the other two**:-

1. $\underline{A}(\underline{r})$ can be written as the **gradient** of a **scalar potential** $\phi(\underline{r})$

$$\boxed{\underline{A}(\underline{r}) \;=\; \underline{\nabla}\,\phi(\underline{r}) \quad \text{with} \quad \phi(\underline{r}) \;=\; \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}')\cdot\underline{dr}'}$$

where $\underline{r}_0$ is some arbitrary fixed point in R.

2. (a) $\oint_C \underline{A}(\underline{r}')\cdot\underline{dr}' \;=\; 0$, where C is any **closed** curve in R

   (b) $\phi(\underline{r}) \equiv \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}')\cdot\underline{dr}'$ does not depend on the path between $\underline{r}_0$ and $\underline{r}$.

3. $\underline{\nabla}\times\underline{A}(\underline{r}) \;=\; 0$ for all points $\underline{r} \in R$

Proof that (2) implies (1)

Consider two neighbouring points $\underline{r}$ and $\underline{r} + \underline{dr}$, define the potential as before

$$\phi(\underline{r}) \;=\; \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}')\cdot\underline{dr}'$$

and define $d\phi$ as

$$\begin{aligned}
d\phi(\underline{r}) \;&=\; \phi(\underline{r}+\underline{dr}) \;-\; \phi(\underline{r}) = \left\{\int_{\underline{r}_0}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' \;-\; \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}')\cdot\underline{dr}'\right\} \quad \text{(by definition)}\\
&=\; \left\{\int_{\underline{r}_0}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' \;+\; \int_{\underline{r}}^{\underline{r}_0} \underline{A}(\underline{r}')\cdot\underline{dr}'\right\} \quad \text{(swapped limits on 2nd } \textstyle\int\text{)}\\
&=\; \int_{\underline{r}}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' \quad \text{(combined integrals using \textbf{path independence})}\\
&=\; \underline{A}(\underline{r})\cdot\underline{dr} \quad \text{(for infinitesimal } \underline{dr}\text{)}
\end{aligned}$$

But, by Taylor's theorem, we also have

$$d\phi(\underline{r}) \;=\; \frac{\partial\phi(\underline{r})}{\partial x_i}\,dx_i \;=\; \underline{\nabla}\,\phi(\underline{r})\cdot\underline{dr}$$

Comparing the two different equations for $d\phi(\underline{r})$, which hold for all $\underline{dr}$, we deduce

$$\underline{A}(\underline{r}) \;=\; \underline{\nabla}\,\phi(\underline{r})$$

Thus we have shown that **path independence** implies the existence of a scalar potential ϕ for the vector field $\underline{A}$. (Also path independence implies 2(a)).

Proof that (1) implies (3) (the easy bit!)

$$\underline{A} = \underline{\nabla}\,\phi \quad \Rightarrow \quad \underline{\nabla} \times \underline{A} = \underline{\nabla} \times \left(\underline{\nabla}\,\phi\right) \equiv 0$$

because curl (grad ϕ) is identically zero (ie it is zero for *any* scalar field ϕ).

Proof that (3) implies (2): (the hard bit!)
We defer the proof until we have met Stokes' theorem in a few lectures time.

Terminology: A vector field is

- **irrotational** if $\underline{\nabla} \times \underline{A}(\underline{r}) = 0$.
- **conservative** if $\underline{A}(\underline{r}) = \underline{\nabla}\phi$.
- For simply connected regions we have shown irrotational and conservative are synonymous. But note that for a multiply connected region this is not the case.

Note: $\phi(\underline{r})$ is only determined up to a **constant**: if $\psi = \phi + constant$ then $\underline{\nabla}\psi = \underline{\nabla}\phi$ and ψ can equally well serve as a potential. The freedom in the constant corresponds to the freedom in choosing $\underline{r}_0$ to calculate the potential. Equivalently the absolute value of a scalar potential has no meaning, only **potential differences** are significant.

4.2.2 Finding Scalar Potentials

We have shown that the scalar potential $\phi(\underline{r})$ for a *conservative* vector field $\underline{A}(\underline{r})$ can be constructed from a line integral which is *independent* of the path of integration between the endpoints. Therefore, a convenient way of evaluating such integrals is to integrate along a **straight line** between the points $\underline{r}_0$ and $\underline{r}$. Choosing $\underline{r}_0 = 0$, we can write this integral in parametric form as follows:

$$\underline{r}' = \lambda\,\underline{r} \quad \text{where} \quad \{0 \leq \lambda \leq 1\} \quad \text{so} \quad \underline{dr}' = d\lambda\,\underline{r} \quad \text{and therefore}$$

$$\phi(\underline{r}) = \int_{\lambda=0}^{\lambda=1} \underline{A}(\lambda\,\underline{r}) \cdot (d\lambda\,\underline{r})$$

Example 1: Let $\underline{A}(\underline{r}) = (\underline{a} \cdot \underline{r})\,\underline{a}$ where $\underline{a}$ is a constant vector.

It is easy to show that $\underline{\nabla} \times ((\underline{a} \cdot \underline{r})\, \underline{a}) = 0$ (tutorial). Thus

$$\begin{aligned} \phi(\underline{r}) &= \int_0^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' = \int_0^{\underline{r}} (\underline{a} \cdot \underline{r}')\, \underline{a} \cdot \underline{dr}' \\ &= \int_0^1 (\underline{a} \cdot \lambda\, \underline{r})\, \underline{a} \cdot (d\lambda\, \underline{r}) = (\underline{a} \cdot \underline{r})^2 \int_0^1 \lambda\, d\lambda \\ &= \frac{1}{2}(\underline{a} \cdot \underline{r})^2 \end{aligned}$$

Note: Always check that your $\phi(\underline{r})$ satisfies $\underline{A}(\underline{r}) \;=\; \underline{\nabla}\, \phi(\underline{r})$!

Example 2: Let $\underline{A}(\underline{r}) \;=\; 2\,(\underline{a} \cdot \underline{r})\, \underline{r} \,+\, r^2\, \underline{a}$ where $\underline{a}$ is a constant vector.
It is straightforward to show that $\underline{\nabla} \times \underline{A} = 0$. Thus

$$\begin{aligned} \phi(\underline{r}) &= \int_0^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; \int_0^1 \underline{A}(\lambda\, \underline{r}) \cdot (d\lambda\, \underline{r}) \\ &= \int_0^1 \Big[2\,(\underline{a} \cdot \lambda\, \underline{r})\, \lambda\, \underline{r} \,+\, \lambda^2 r^2\, \underline{a} \Big] \cdot (d\lambda\, \underline{r}) \\ &= \Big[2\,(\underline{a} \cdot \underline{r})\, \underline{r} \cdot \underline{r} \,+\, r^2\, (\underline{a} \cdot \underline{r}) \Big] \int_0^1 \lambda^2\, d\lambda \\ &= r^2\, (\underline{a} \cdot \underline{r}) \end{aligned}$$

Example 2 (revisited): Again, let $\underline{A}(\underline{r}) \;=\; 2\,(\underline{a} \cdot \underline{r})\, \underline{r} \,+\, r^2\, \underline{a}$ here $\underline{a}$ is a constant vector.

$$\underline{A}(\underline{r}) \;=\; 2\,(\underline{a} \cdot \underline{r})\, \underline{r} \,+\, r^2\, \underline{a} \;=\; (\underline{a} \cdot \underline{r})\underline{\nabla} r^2 \,+\, r^2\, \underline{\nabla}(\underline{a} \cdot \underline{r}) \;=\; \underline{\nabla}\Big((\underline{a} \cdot \underline{r})\, r^2 \,+\, \mathrm{const} \Big)$$

in agreement with what we had before if we choose const $=$ 0.
While this method is not as systematic as Method 1, it can be quicker if you spot the trick!

4.2.3 Conservative forces: conservation of energy

Let us now see how the name *conservative field* arises. Consider a vector field $\underline{F}(\underline{r})$ corresponding to the only force acting on some test particle of mass m. We will show that for a conservative force (where we can write $\underline{F} = -\underline{\nabla} V$) the total energy is **constant** in time.

Proof: The particle moves under the influence of Newton's Second Law:

$$m\ddot{\underline{r}} = \underline{F}(\underline{r}).$$

Consider a small displacement $\underline{dr}$ along the path taking time dt. Then

$$m\ddot{\underline{r}} \cdot \underline{dr} = \underline{F}(\underline{r}) \cdot \underline{dr} = -\underline{\nabla} V(\underline{r}) \cdot \underline{dr}.$$

Integrating this expression along the path from $\underline{r}_A$ at time $t = t_A$ to $\underline{r}_B$ at time $t = t_B$ yields

$$m \int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}} \cdot \underline{dr} = - \int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla} V(\underline{r}) \cdot \underline{dr}.$$

We can simplify the left-hand side of this equation to obtain

$$m \int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}} \cdot \underline{dr} = m \int_{t_A}^{t_B} \ddot{\underline{r}} \cdot \dot{\underline{r}}\, dt = m \int_{t_A}^{t_B} \tfrac{1}{2} \frac{d}{dt} \dot{\underline{r}}^2 dt = \tfrac{1}{2} m [v_B^2 - v_A^2],$$

where v_A and v_B are the magnitudes of the velocities at points A and B respectively.

The right-hand side simply gives

$$- \int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla} V(\underline{r}) \cdot \underline{dr} = - \int_{\underline{r}_A}^{\underline{r}_B} dV = V_A - V_B$$

where V_A and V_B are the values of the potential V at $\underline{r}_A$ and $\underline{r}_B$, respectively. Therefore

$$\frac{1}{2} m v_A^2 + V_A = \frac{1}{2} m v_B^2 + V_B$$

and the total energy $E = \frac{1}{2} m v^2 + V$ is **conserved**, i.e. *constant in time*.

Newtonian gravity and the *electrostatic force* are both conservative. *Frictional forces* are not conservative; energy is dissipated and work is done in traversing a closed path. In general, time-dependent forces are not conservative.

end of lecture 17

4.2.4 Physical Examples of Conservative Forces

Newtonian Gravity and the *electrostatic force* are both conservative. *Frictional forces* are not conservative; energy is dissipated and work is done in traversing a closed path. In general, time-dependent forces are not conservative.

The foundation of Newtonian Gravity is **Newton's Law of Gravitation**. The force $\underline{F}$ on a particle of mass m_1 at $\underline{r}$ due to a particle of mass m at the origin is given by

$$\underline{F} = -G\, m\, m_1 \frac{\hat{\underline{r}}}{r^2}$$

where $G = 6.672\,59(85) \times 10^{-11}\, Nm^2 kg^2$ is Newton's Gravitational Constant.

The **gravitational field** $\underline{G}(\underline{r})$ (due to the mass at the origin) is formally defined as

$$\underline{G}(\underline{r}) = \lim_{m_1 \to 0} \frac{\underline{F}(\underline{r})}{m_1}.$$

so that the gravitational field due to the test mass m_1 can be ignored. The **gravitational potential** can be obtained by spotting the direct integration for $\underline{G} = -\underline{\nabla}\phi$

$$\phi = -\frac{Gm}{r}\ .$$

Alternatively, to calculate by a line integral choose $\underline{r}_0 = \infty$ then

$$\begin{aligned}\phi(\underline{r}) &= -\int_{\infty}^{\underline{r}} \underline{G}(\underline{r}')\cdot d\underline{r}' = -\int_{\infty}^{1} \underline{G}(\lambda\underline{r})\cdot d\lambda\underline{r} \\ &= \int_{\infty}^{1} \frac{Gm\,(\hat{\underline{r}}\cdot\underline{r})}{r^2}\frac{d\lambda}{\lambda^2} = -\frac{Gm}{r}\end{aligned}$$

NB In this example the vector field $\underline{G}$ is singular at the origin $\underline{r} = 0$. This implies we have to exclude the origin and it is not possible to obtain the scalar potential at $\underline{r}$ by integration along a path from the origin. Instead we integrate from infinity, which in turn means that the gravitational potential at infinity is zero.

NB Since $\underline{F} = m_1\underline{G} = -\underline{\nabla}(m_1\phi)$ the potential energy of the mass m_1 is $V = m_1\phi$. The distinction (a convention) between potential and potential energy is a common source of confusion.

Electrostatics: Coulomb's Law states that the force $\underline{F}$ on a particle of charge q_1 at $\underline{r}$ in the electric field $\underline{E}$ due to a particle of charge q at the origin is given by

$$\underline{F} = q_1\underline{E} = \frac{q_1\, q}{4\pi\epsilon_0}\frac{\hat{\underline{r}}}{r^2}$$

where $\epsilon_0 = 8.854\,187\,817\cdots\times 10^{-12}\,C^2N^{-1}m^{-2}$ is the **Permitivity of Free Space** and the 4π is conventional. More strictly,

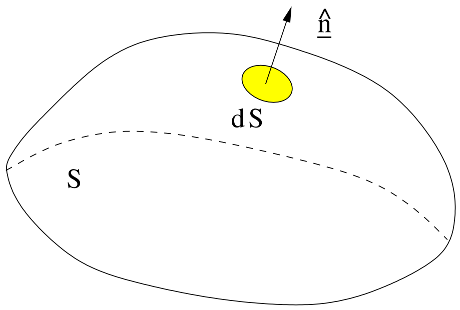
$$\underline{E}(\underline{r}) = \lim_{q_1\to 0}\frac{\underline{F}(\underline{r})}{q_1}.$$

The **electrostatic potential** is taken as $\phi = 1/(4\pi\epsilon_0 r)$ (obtained by integrating $E = -\underline{\nabla}\phi$ from infinity to $\underline{r}$) and the potential energy of a charge q_1 in the electric field is $V = q_1\phi$.

Note that mathematically electrostatics and gravitation are very similar, the only real difference being that gravity between two masses is always attractive, whereas like charges repel.

end of add-on to lecture 17

4.3 Surface Integrals



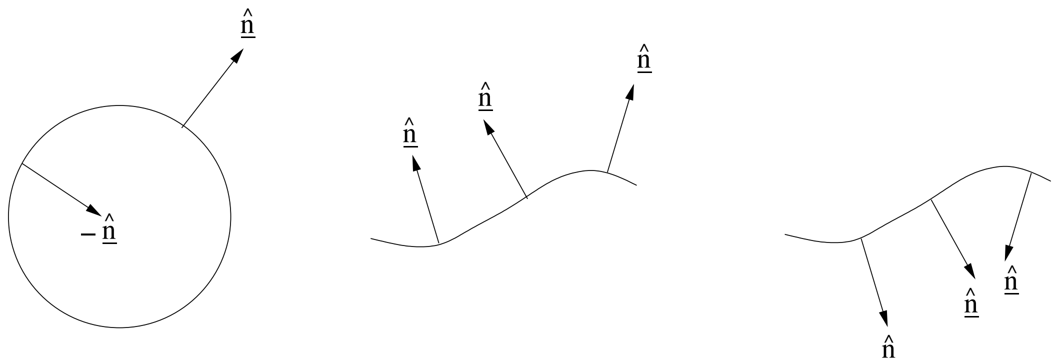
Let S be a two-sided surface in ordinary three-dimensional space as shown. If an infinitesimal element of surface with (scalar) area dS has unit normal $\underline{\hat{n}}$, then the infinitesimal **vector element of area** is *defined* by:-

$$\boxed{\underline{dS} \;=\; \underline{\hat{n}}\, dS}$$

Example: if S lies in the (x, y) plane, then $\underline{dS} = \underline{e}_3\, dx\, dy$ in Cartesian coordinates.

Physical interpretation: $\underline{dS} \cdot \underline{\hat{a}}$ gives the projected (scalar) element of area onto the plane with unit normal $\underline{\hat{a}}$.

For **closed** surfaces (eg, a sphere) we *choose* $\underline{\hat{n}}$ to be the **outward normal**. For **open** surfaces, the sense of $\underline{\hat{n}}$ is arbitrary — except that it is chosen in the same sense for all elements of the surface. See *Bourne & Kendall 5.5* for further discussion of surfaces.



If $\underline{A}(\underline{r})$ is a vector field defined on S, we define the (normal) **surface integral**

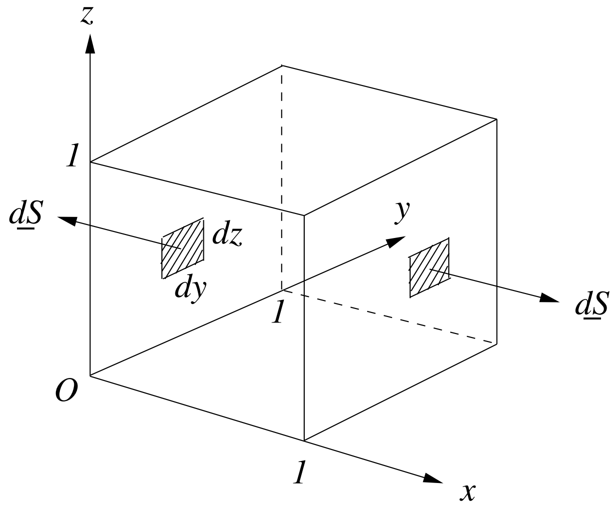
$$\int_S \underline{A} \cdot \underline{dS} \;=\; \int_S \left(\underline{A} \cdot \underline{\hat{n}}\right)\, dS \;=\; \lim_{\substack{m \to \infty \\ \delta S \to 0}} \sum_{i=1}^{m} \left(\underline{A}(\underline{r}^{\,i}) \cdot \underline{\hat{n}}^{\,i}\right)\, \delta S^i$$

where we have formed the Riemann sum by dividing the surface S into m small areas, the ith area having vector area $\underline{\delta S}^{\,i}$. Clearly, the quantity $\underline{A}(\underline{r}^{\,i}) \cdot \underline{\hat{n}}^{\,i}$ is the component of $\underline{A}$ *normal* to the surface at the point $\underline{r}^{\,i}$

- We use the notation $\int_S \underline{A} \cdot \underline{dS}$ for both *open* and *closed* surfaces. Sometimes the integral over a *closed* surface is denoted by $\oint_S \underline{A} \cdot \underline{dS}$ (*not* used here).

- Note that the integral over S is a **double integral** in each case. Hence surface integrals are sometimes denoted by $\int\int_S \underline{A} \cdot \underline{dS}$ (*not* used here).

Example: Let S be the surface of a unit cube (S = sum over **all six faces**).



On the front face, parallel to the (y, z) plane, at $x = 1$, we have

$$\underline{dS} \;=\; \underline{\hat{n}}\, dS \;=\; \underline{e}_1\, dy\, dz$$

On the back face at $x = 0$ in the (y, z) plane, we have

$$\underline{dS} \;=\; \underline{\hat{n}}\, dS \;=\; -\underline{e}_1\, dy\, dz$$

In each case, the unit normal $\underline{\hat{n}}$ is an *outward* normal because S is a *closed* surface.

If $\underline{A}(\underline{r})$ is a vector field, then the integral $\int_S \underline{A} \cdot \underline{dS}$ over the front face shown is

$$\int_{z=0}^{z=1} \int_{y=0}^{y=1} \underline{A} \cdot \underline{e}_1\, dy\; dz \;=\; \int_{z=0}^{z=1} \int_{y=0}^{y=1} A_1 \Big|_{x=1} dy\; dz$$

The integral over y and z is an ordinary double integral over a square of side 1. The integral over the back face is

$$-\int_{z=0}^{z=1} \int_{y=0}^{y=1} \underline{A} \cdot \underline{e}_1\, dy\; dz \;=\; -\int_{z=0}^{z=1} \int_{y=0}^{y=1} A_1 \Big|_{x=0} dy\; dz$$

The total integral is the sum of contributions from all 6 faces.

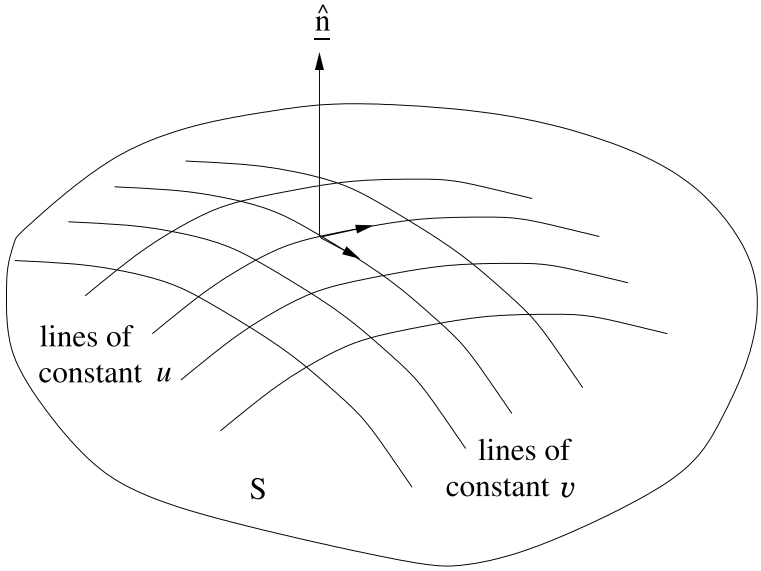
4.3.1 Parametric form of the surface integral

Suppose the points on a surface S are defined by **two** real parameters u and v:-

$\underline{r} = \underline{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ then

- the lines $\underline{r}(u, v)$ for fixed u, variable v, and
- the lines $\underline{r}(u, v)$ for fixed v, variable u

are **parametric lines** and form a **grid** on the surface S as shown.



If we change u and v by du and dv respectively, then $\underline{r}$ changes by $\underline{dr}$:-

$$\underline{dr} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv$$

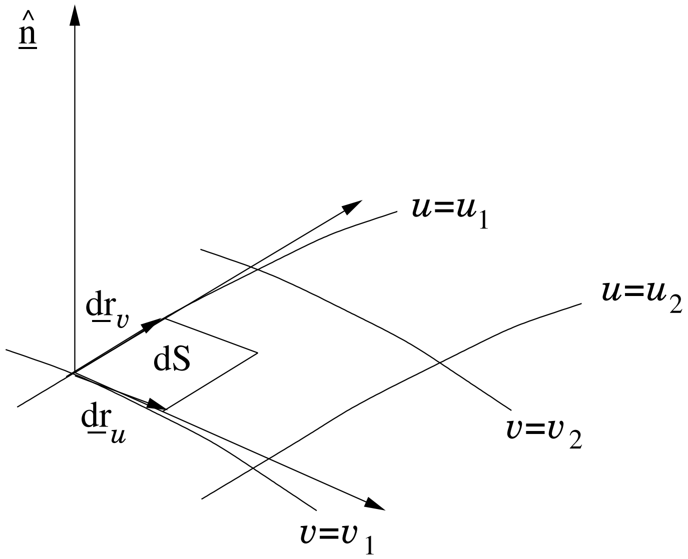
Along the curves $v =$ constant, we have $dv = 0$, and so $\underline{dr}$ is simply:-

$$\underline{dr}_u = \frac{\partial \underline{r}}{\partial u} du$$

where $\frac{\partial \underline{r}}{\partial u}$ is a vector which is tangent to the surface, and tangent to the lines $v =$ const. Similarly, for $u =$ constant, we have

$$\underline{dr}_v = \frac{\partial \underline{r}}{\partial v} dv$$

so $\frac{\partial \underline{r}}{\partial v}$ is tangent to lines $u =$ constant.



We can therefore construct a **unit vector** $\underline{\hat{n}}$, **normal** to the surface at $\underline{r}$:-

$$\underline{\hat{n}} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \Big/ \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right|$$

The vector element of area, $\underline{dS}$, has magnitude equal to the area of the infinitesimal parallelogram shown, and points in the direction of $\underline{\hat{n}}$, therefore we can write

$$\underline{dS} = \underline{dr}_u \times \underline{dr}_v = \left(\frac{\partial \underline{r}}{\partial u}\, du \right) \times \left(\frac{\partial \underline{r}}{\partial v}\, dv \right) = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right) du\, dv$$

$$\boxed{\underline{dS} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right) du\, dv}$$

Finally, our integral is parameterised as

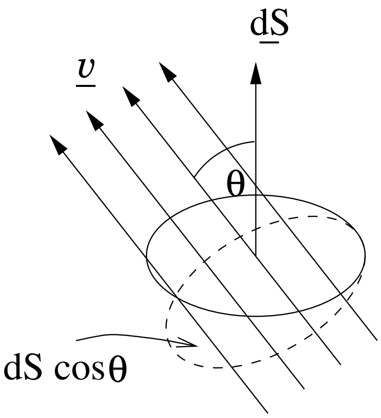
$$\boxed{\int_S \underline{A} \cdot \underline{dS} = \int_v \int_u \underline{A} \cdot \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right) du\, dv}$$

Note: We use two integral signs when writing surface integrals in terms of **explicit** parameters u and v. The limits for the integrals over u and v must be chosen appropriately for the surface.

end of lecture 18

4.4 More on Surface and Volume Integrals

4.4.1 The Concept of Flux



Let $\underline{v}(\underline{r})$ be the velocity at a point $\underline{r}$ in a moving fluid. In a small region, where $\underline{v}$ is approximately constant, the **volume** of fluid crossing the element of vector area $\underline{dS} = \underline{\hat{n}}\, dS$ in time dt is

$$\big(|\underline{v}|\ dt\big)\ \ (dS\ \cos\theta)\ \ =\ \ \big(\underline{v}\cdot\underline{dS}\big)\ dt$$

since the area *normal* to the direction of flow is $\underline{\hat{v}}\cdot\underline{dS} = dS\ \cos\theta$.

Therefore

$$\begin{aligned} \underline{v}\cdot\underline{dS} &= \textbf{volume per unit time} \text{ of fluid crossing } \underline{dS} \\ \text{hence}\quad \int_S \underline{v}\cdot\underline{dS} &= \textbf{volume per unit time} \text{ of fluid crossing a finite surface } S \end{aligned}$$

More generally, for a vector field $\underline{A}(\underline{r})$:

The surface integral $\int_S \underline{A}\cdot\underline{dS}$ is called the **flux** of $\underline{A}$ through the surface S.

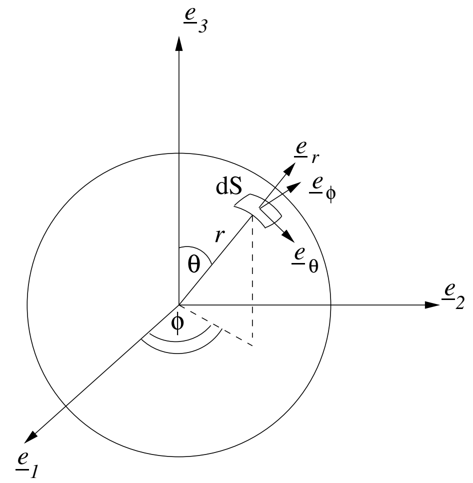
The concept of flux is useful in many different contexts e.g. flux of molecules in an gas; electromagnetic flux etc

Example: Let S be the surface of sphere $x^2 + y^2 + z^2 = a^2$.

Find $\underline{\hat{n}}$, $\underline{dS}$ and evaluate the total flux of the vector field $\underline{A} = \underline{\hat{r}}/r^2$ out of the sphere.

An arbitrary point $\underline{r}$ on S may be parameterised by spherical polar co-ordinates θ and ϕ

$$\begin{aligned} \underline{r} &= a\sin\theta\cos\phi\,\underline{e}_1 + a\sin\theta\sin\phi\,\underline{e}_2 + a\cos\theta\,\underline{e}_3 \qquad \{0\le\theta\le\pi,\ 0\le\phi\le 2\pi\} \\ \text{so}\quad \frac{\partial\underline{r}}{\partial\theta} &= a\cos\theta\cos\phi\,\underline{e}_1 + a\cos\theta\sin\phi\,\underline{e}_2 - a\sin\theta\,\underline{e}_3 \\ \text{and}\quad \frac{\partial\underline{r}}{\partial\phi} &= -a\sin\theta\sin\phi\,\underline{e}_1 + a\sin\theta\cos\phi\,\underline{e}_2 + 0\,\underline{e}_3 \end{aligned}$$



Therefore

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a\,\cos\theta\cos\phi & a\,\cos\theta\sin\phi & -a\,\sin\theta \\ -a\sin\theta\sin\phi & +a\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= a^2\,\sin^2\theta\cos\phi\,\underline{e}_1 \,+\, a^2\,\sin^2\theta\sin\phi\,\underline{e}_2 \,+\, a^2\,\sin\theta\cos\theta\,\left[\cos^2\phi+\sin^2\phi\right]\underline{e}_3 \\
&= a^2\sin\theta\,(\sin\theta\cos\phi\,\underline{e}_1+\sin\theta\sin\phi\,\underline{e}_2+\cos\theta\,\underline{e}_3) \\
&= a^2\sin\theta\;\hat{\underline{r}} \\
\hat{\underline{n}} &= \hat{\underline{r}} \\
\underline{dS} &= \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\,d\theta\,d\phi \;=\; a^2\sin\theta d\theta\,d\phi\,\hat{\underline{r}}
\end{aligned}$$

On the surface S, $r = a$ and the vector field $\underline{A}(\underline{r}) \;=\; \hat{\underline{r}}/a^2$. Thus the flux of $\underline{A}$ is

$$\int_S \underline{A}\cdot\underline{dS} = \int_0^{\pi}\sin\theta d\theta\,\int_0^{2\pi} d\phi = 4\pi$$

Spherical basis: The normalised vectors (shown in the figure)

$$\underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta}\bigg/\left|\frac{\partial \underline{r}}{\partial \theta}\right| \quad;\quad \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi}\bigg/\left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad;\quad \underline{e}_r = \hat{\underline{r}}$$

form an orthonormal set. This is the basis for spherical polar co-ordinates and is an example of a non-Cartesian basis since the $\underline{e}_\theta, \underline{e}_\phi, \underline{e}_r$ depend on position $\underline{r}$.

4.4.2 Other Surface Integrals

If $f(\underline{r})$ is a scalar field, a scalar surface integral is of the form

$$\int_S f\,dS$$

For example the **surface area** of the surface S is

$$\int_S dS \;=\; \int_S |\underline{dS}| \;=\; \int_v\int_u \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right| \, du\, dv$$

We may also define vector surface integrals:-

$$\int_S f\ \underline{dS} \qquad \int_S \underline{A}\ dS \qquad \int_S \underline{A} \times \underline{dS}$$

Each of these is a double integral, and is evaluated in a similar fashion to the scalar integrals, the result being a vector in each case.

The **vector area** of a surface is defined as $\underline{S} = \displaystyle\int_S \underline{dS}$. For a closed surface this is always zero.

Example: the vector area of an (open) hemisphere (see 16.1) of radius a is found using spherical polars to be

$$\underline{S} = \int_S \underline{dS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} a^2 \sin\theta \underline{e_r}\ d\theta\, d\phi\ .$$

Using $\underline{e_r} = \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3$ we obtain

$$\begin{aligned}
\underline{S} \;&=\; \underline{e}_1\, a^2 \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \cos\phi d\phi \;+\; \underline{e}_2\, a^2 \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi d\phi \\
&\quad + \underline{e}_3\, a^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \\
&=\; 0 + 0 + \underline{e}_3 \pi a^2
\end{aligned}$$

The vector surface of the full sphere is zero since the contributions from upper and lower hemispheres cancel; also the vector area of a *closed* hemisphere is zero since the vector area of the bottom face is $-\underline{e}_3\pi a^2$.

4.4.3 Parametric form of Volume Integrals

We have already met and revised volume integrals in 16.1. Conceptually volume integrals are simpler than line and surface integrals because the elemental volume dV is a scalar quantity.

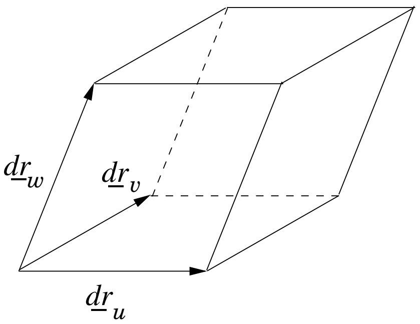
Here we discuss the *parametric* form of volume integrals. Suppose we can write $\underline{r}$ in terms of three real parameters u, v and w, so that $\underline{r} = \underline{r}(u, v, w)$. If we make a small change in each of these parameters, then $\underline{r}$ changes by

$$\underline{dr} \;=\; \frac{\partial \underline{r}}{\partial u}\, du \;+\; \frac{\partial \underline{r}}{\partial v}\, dv \;+\; \frac{\partial \underline{r}}{\partial w}\, dw$$

Along the curves $\{v = \text{constant},\, w = \text{constant}\}$, we have $dv = 0$ and $dw = 0$, so $\underline{dr}$ is simply:-

$$\underline{dr}_u \;=\; \frac{\partial \underline{r}}{\partial u}\, du$$

with $\underline{dr}_v$ and $\underline{dr}_w$ having analogous definitions.



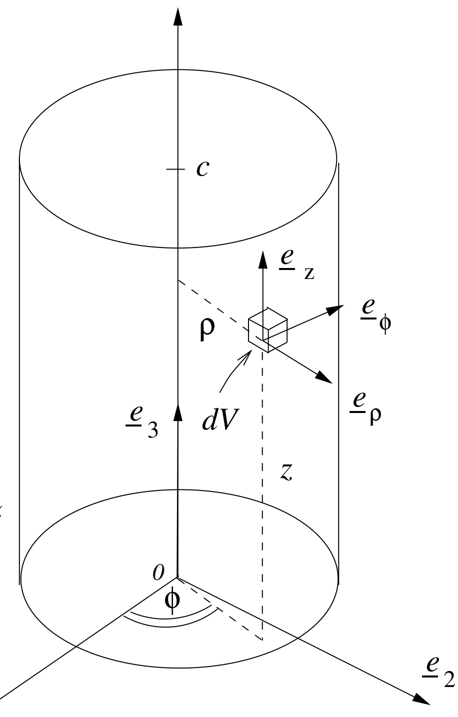
The vectors $\underline{dr}_u$, $\underline{dr}_v$ and $\underline{dr}_w$ form the sides of an infinitesimal parallelepiped of volume

$$dV \;=\; \left| \underline{dr}_u \cdot \underline{dr}_v \times \underline{dr}_w \right|$$

$$\boxed{dV = \left| \frac{\partial \underline{r}}{\partial u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w} \right| \; du\, dv\, dw}$$

Example: Consider a circular cylinder of radius a, height c. We can parameterise $\underline{r}$ using cylindrical polar coordinates. Within the cylinder, we have

$$\underline{r} = \rho \cos\phi\, \underline{e}_1 \,+\, \rho \sin\phi\, \underline{e}_2 \,+\, z \underline{e}_3 \quad \{0 \le \rho \le a,\; 0 \le \phi \le 2\pi,\; 0 \le z \le c\}$$



$$\text{Thus} \qquad \frac{\partial \underline{r}}{\partial \rho} \;=\; \cos\phi\, \underline{e}_1 \,+\, \sin\phi\, \underline{e}_2$$

$$\frac{\partial \underline{r}}{\partial \phi} \;=\; -\rho\, \sin\phi\, \underline{e}_1 \,+\, \rho\, \cos\phi\, \underline{e}_2$$

$$\frac{\partial \underline{r}}{\partial z} \;=\; \underline{e}_3$$

$$\text{and so} \qquad dV \;=\; \left| \frac{\partial \underline{r}}{\partial \rho} \cdot \frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial z} \right| \; d\rho\, d\phi\, dz \;=\; \rho\, d\rho\, d\phi\, dz$$

The **volume** of the cylinder is

$$\int_V dV \;=\; \int_{z=0}^{z=c} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=a} \rho\, d\rho\, d\phi\, dz \;=\; \pi\, a^2 c.$$

Cylindrical basis: the normalised vectors (shown on the figure) form a non-Cartesian basis where

$$\underline{e}_\rho = \frac{\partial \underline{r}}{\partial \rho} \Big/ \left| \frac{\partial \underline{r}}{\partial \rho} \right| \quad ; \quad \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi} \Big/ \left| \frac{\partial \underline{r}}{\partial \phi} \right| \quad ; \quad \underline{e}_z = \frac{\partial \underline{r}}{\partial z} \Big/ \left| \frac{\partial \underline{r}}{\partial z} \right|$$

4.5 The Divergence Theorem

4.5.1 Integral Definition of Divergence

If $\underline{A}$ is a vector field in the region R, and P is a point in R, then the divergence of $\underline{A}$ at P may be **defined** by

$$\operatorname{div} \underline{A} = \lim_{V \to 0} \frac{1}{V} \int_S \underline{A} \cdot \underline{dS}$$

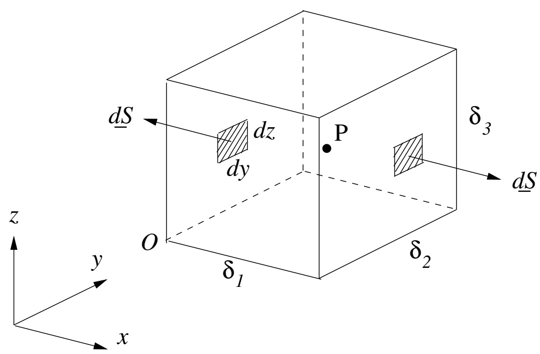
where S is a **closed** surface in R which encloses the volume V. The limit must be taken so that the point P is within V.

This definition of div $\underline{A}$ is **basis independent**.

We now prove that our original definition of *div* is recovered in Cartesian co-ordinates

Let P be a point with Cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangular block of size $\delta_1 \times \delta_2 \times \delta_3$, so its volume is $\delta V = \delta_1\, \delta_2\, \delta_3$.

- On the **front** face of the block, orthogonal to the x axis at $x = x_0 + \delta_1/2$ we have *outward* normal $\underline{\hat{n}} = \underline{e}_1$ and so $\underline{dS} = \underline{e}_1\, dy\, dz$
- On the **back** face of the block orthogonal to the x axis at $x = x_0 - \delta_1/2$ we have *outward* normal $\underline{\hat{n}} = -\underline{e}_1$ and so $\underline{dS} = -\underline{e}_1\, dy\, dz$



Hence $\underline{A} \cdot \underline{dS} = \pm\, A_1\, dy\, dz$ on these two faces. Let us denote the two surfaces orthogonal to the $\underline{e}_1$ axis by S_1.

The contribution of these two surfaces to the integral $\int_S \underline{A} \cdot \underline{dS}$ is given by

$$\begin{aligned}
\int_{S_1} \underline{A} \cdot \underline{dS} &= \int_z \int_y \Big\{ A_1(x_0 + \delta_1/2, y, z) \; - \; A_1(x_0 - \delta_1/2, y, z) \Big\} \, dy \, dz \\
&= \int_z \int_y \Bigg\{ \bigg[A_1(x_0, y, z) \; + \; \frac{\delta_1}{2} \, \frac{\partial A_1(x_0, y, z)}{\partial x} \; + \; O(\delta_1^2) \bigg] \\
&\qquad\qquad - \bigg[A_1(x_0, y, z) \; - \; \frac{\delta_1}{2} \, \frac{\partial A_1(x_0, y, z)}{\partial x} \; + \; O(\delta_1^2) \bigg] \Bigg\} \, dy \, dz \\
&= \int_z \int_y \delta_1 \, \frac{\partial A_1(x_0, y, z)}{\partial x} \, dy \, dz
\end{aligned}$$

where we have dropped terms of $O(\delta_1^2)$ in the Taylor expansion of A_1 about (x_0, y, z).

So

$$\frac{1}{\delta V} \int_{S_1} \underline{A} \cdot \underline{dS} = \frac{1}{\delta_2 \, \delta_3} \int_z \int_y \frac{\partial A_1(x_0, y, z)}{\partial x} \, dy \, dz$$

As we take the limit $\delta_1, \delta_2, \delta_3 \to 0$ the integral tends to $\frac{\partial A_1(x_0, y_0, z_0)}{\partial x} \, \delta_2 \, \delta_3$ and we obtain

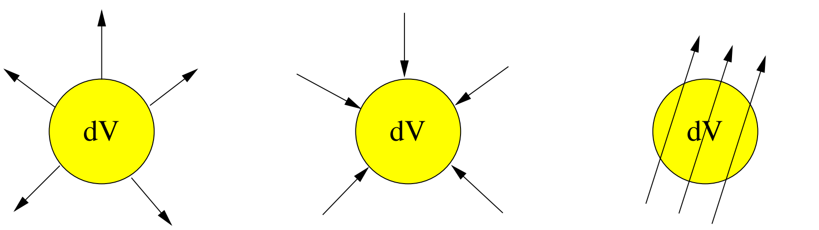
$$\lim_{\delta V \to 0} \frac{1}{\delta V} \int_{S_1} \underline{A} \cdot \underline{dS} \; = \; \frac{\partial A_1(x_0, y_0, z_0)}{\partial x}$$

With similar contributions from the other 4 faces, we find

$$\text{div} \, \underline{A} \; = \; \frac{\partial A_1}{\partial x} \; + \; \frac{\partial A_2}{\partial y} \; + \; \frac{\partial A_3}{\partial z} \; = \; \underline{\nabla} \cdot \underline{A}$$

in agreement with our original definition in Cartesian co-ordinates.

Note that the integral definition gives an intuitive understanding of the divergence in terms of net flux leaving a small volume around a point $\underline{r}$. **In pictures:** for a small volume dV



div $\underline{A} > 0$ div $\underline{A} < 0$ div $\underline{A} = 0$ (flux in = flux out)

4.5.2 The Divergence Theorem (Gauss's Theorem)

If $\underline{A}$ is a vector field in a volume V, and S is the closed surface bounding V, then

$$\boxed{\int_V \underline{\nabla} \cdot \underline{A}\, dV \;=\; \int_S \underline{A} \cdot \underline{dS}}$$

Proof : We derive the divergence theorem by making use of the integral definition of div $\underline{A}$

$$\text{div}\, \underline{A} \;=\; \lim_{\text{V}\to 0} \frac{1}{\text{V}} \int_{\text{S}} \underline{A} \cdot \underline{dS}.$$

Since this **definition** of div $\underline{A}$ is valid for volumes of arbitrary shape, we can build a smooth surface S from a large number, N, of blocks of volume ΔV^i and surface ΔS^i. We have

$$\text{div}\, \underline{A}(\underline{r}^i) \;=\; \frac{1}{\Delta V^i} \int_{\Delta S^i} \underline{A} \cdot \underline{dS} \,+\, (\epsilon^i)$$

where $\epsilon^i \to 0$ as $\Delta V^i \to 0$. Now multiply both sides by ΔV^i and sum over all i

$$\sum_{i=1}^{N} \text{div}\, \underline{A}(\underline{r}^i)\, \Delta V^i \;=\; \sum_{i=1}^{N} \int_{\Delta S^i} \underline{A} \cdot \underline{dS} \,+\, \sum_{i=1}^{N} \epsilon^i\, \Delta V^i$$

On rhs the contributions from surface elements *interior* to S cancel. This is because where two blocks touch, the outward normals are in *opposite* directions, implying that the contributions to the respective integrals cancel.

Taking the limit $N \to \infty$ we have, as claimed,

$$\int_V \underline{\nabla} \cdot \underline{A}\, dV \;=\; \int_S \underline{A} \cdot \underline{dS}\,.$$

For an elegant alternative proof see *Bourne & Kendall 6.2*

4.6 The Continuity Equation

Consider a fluid with density field $\rho(\underline{r})$ and velocity field $\underline{v}(\underline{r})$. We have seen previously that the volume flux (volume per unit time) flowing across a surface is given by $\int_S \underline{v} \cdot \underline{dS}$. The corresponding mass flux (mass per unit time) is given by

$$\int_S \rho \underline{v} \cdot \underline{dS} \equiv \int_S \underline{J} \cdot \underline{dS}$$

where $\underline{J} = \rho \underline{v}$ is called the *mass current*.

Now consider a volume V bounded by the *closed* surface S containing no sources or sinks of fluid. Conservation of mass means that the outward mass flux through the surface S must be equal to the rate of decrease of mass contained in the volume V.

$$\int_S \underline{J} \cdot \underline{dS} = -\frac{\partial M}{\partial t}\,.$$

The mass in V may be written as $M = \int_V \rho \, dV$. Therefore we have

$$\frac{\partial}{\partial t} \int_V \rho \, dV + \int_S \underline{J} \cdot \underline{dS} = 0 \; .$$

We now use the divergence theorem to rewrite the second term as a volume integral and we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} \right] dV = 0$$

Now since this holds for arbitrary V we must have that

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \; .}$$

This equation, known as the **continuity equation**, appears in many different contexts since it holds for any *conserved* quantity. Here we considered mass density ρ and mass current $\underline{J}$ of a fluid; but equally it could have been number density of molecules in a gas and current of molecules; electric charge density and electric current vector; thermal energy density and heat current vector; or even more abstract conserved quantities such as probability density!

4.7 Sources and Sinks

Static case: Consider *time independent* behaviour where $\frac{\partial \rho}{\partial t} = 0$. The continuity equation tells us that for the density to be constant in time we must have $\underline{\nabla} \cdot \underline{J} = 0$ so that flux into a point equals flux out.

However if we have a **source** or a **sink** of the field, the divergence is not zero at that point. In general the quantity

$$\frac{1}{V} \int_S \underline{A} \cdot \underline{dS}$$

tells us whether there are sources or sinks of the vector field $\underline{A}$ within V: if V contains

- a **source**, then $\int_S \underline{A} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{A} \, dV > 0$
- a **sink**, then $\int_S \underline{A} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{A} \, dV < 0$

If S contains neither sources nor sinks, then $\int_S \underline{A} \cdot \underline{dS} = 0$.

As an example consider **electrostatics**. You will have learned that electric field lines are conserved and can only start and stop at charges. A positive charge is a source of electric field (i.e. creates a positive flux) and a negative charge is a sink (i.e. absorbs flux or creates a negative flux).

The electric field due to a charge q at the origin is

$$\underline{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\underline{r}}}{r^2}.$$

It is easy to verify that $\underline{\nabla} \cdot \underline{E} = 0$ except at the origin where the field is singular.

The flux integral for this type of field across a sphere (of any radius) around the origin was evaluated in the last lecture and we find the flux out of the sphere as:

$$\int_S \underline{E} \cdot \underline{dS} = \frac{q}{\epsilon_0}$$

Now since $\underline{\nabla} \cdot \underline{E} = 0$ away from the origin the results holds for *any surface enclosing the origin*. Moreover if we have several charges enclosed by S then

$$\int_S \underline{E} \cdot \underline{dS} = \sum_i \frac{q_i}{\epsilon_0}.$$

This recovers *Gauss' Law* of electrostatics.

We can go further and consider a *charge density* of $\rho(\underline{r})$ per unit volume. Then

$$\int_S \underline{E} \cdot \underline{dS} = \int_V \frac{\rho(\underline{r})}{\epsilon_0} dV \ .$$

We can rewrite the lhs using the divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{E} \, dV = \int_V \frac{\rho(\underline{r})}{\epsilon_0} dV \ .$$

Since this must hold for arbitrary V we see

$$\boxed{\underline{\nabla} \cdot \underline{E} = \frac{\rho(\underline{r})}{\epsilon_0}}$$

which holds for all $\underline{r}$ and is one of Maxwell's equations of Electromagnetism.

end of lecture 20

4.8 Examples of the Divergence Theorem

Volume of a body:

Consider the volume of a body:

$$V = \int_V dV$$

Recalling that $\underline{\nabla} \cdot \underline{r} = 3$ we can write

$$V = \frac{1}{3} \int_V \underline{\nabla} \cdot \underline{r} \, dV$$

which using the divergence theorem becomes

$$V = \frac{1}{3} \int_S \underline{r} \cdot \underline{dS}$$

Example: Consider the hemisphere $x^2 + y^2 + z^2 \leq a^2$ centered on $\underline{e}_3$ with bottom face at $z = 0$. Recalling that the divergence theorem holds for a *closed* surface, the above equation for the volume of the hemisphere tells us

$$V = \frac{1}{3} \left[\int_{hemisphere} \underline{r} \cdot \underline{dS} + \int_{bottom} \underline{r} \cdot \underline{dS} \right] .$$

On the bottom face $\underline{dS} = -\underline{e}_3 \, dS$ so that $\underline{r} \cdot \underline{dS} = -z \, dS = 0$ since $z = 0$. Hence the only contribution comes from the (open) surface of the hemisphere and we see that

$$V = \frac{1}{3} \int_{hemisphere} \underline{r} \cdot \underline{dS} .$$

We can evaluate this by using spherical polars for the surface integral. As was derived in lecture 19, for a hemisphere of radius a

$$\underline{dS} = a^2 \, \sin\theta \, d\theta \, d\phi \, \underline{e}_r .$$

On the hemisphere $\underline{r} \cdot \underline{dS} = a^3 \, \sin\theta \, d\theta \, d\phi$ so that

$$\int_S \underline{r} \cdot \underline{dS} = a^3 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 2\pi a^3$$

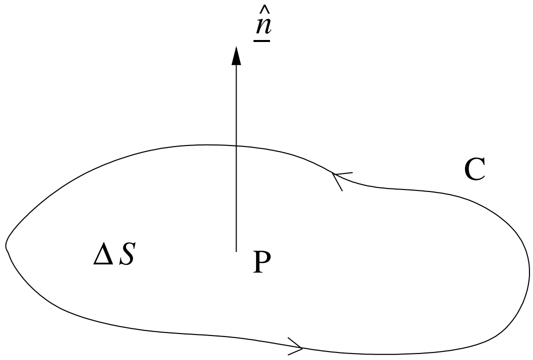
giving the anticipated result

$$V = \frac{2\pi a^3}{3} .$$

4.9 Line Integral Definition of Curl and Stokes' Theorem

4.9.1 Line Integral Definition of Curl

Let ΔS be a small planar surface containing the point P, bounded by a **closed** curve C, with unit normal $\hat{\underline{n}}$ and (scalar) area ΔS. Let $\underline{A}$ be a vector field defined on ΔS.



The component of $\underline{\nabla} \times \underline{A}$ parallel to $\hat{\underline{n}}$ is defined to be

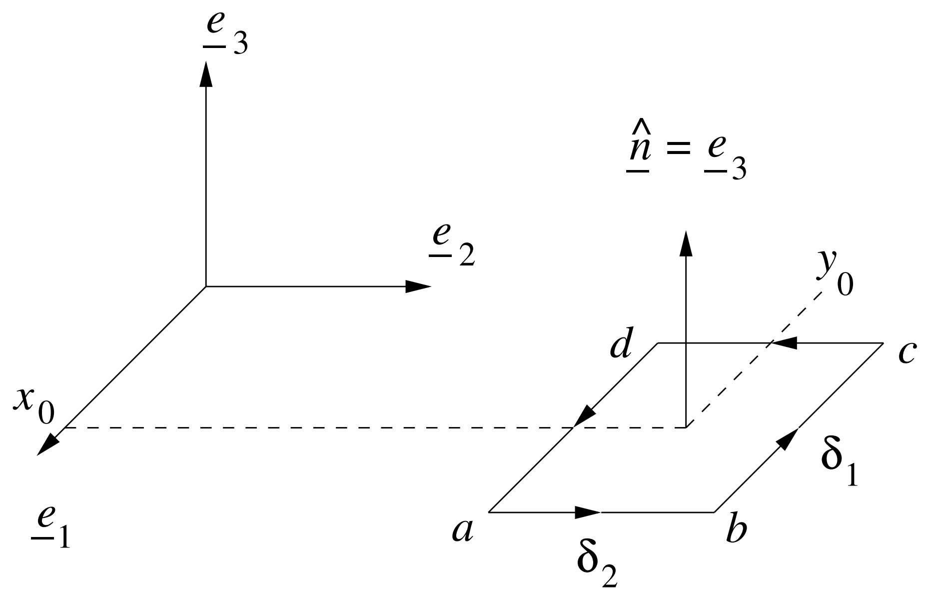
$$\boxed{\underline{\hat{n}}\cdot(\underline{\nabla} \times \underline{A}) \;=\; \lim_{\Delta S\to 0} \frac{1}{\Delta S} \oint_C \underline{A}\cdot\underline{dr}}$$

NB: the integral around C is taken in the right-hand sense with respect to the normal $\underline{\hat{n}}$ to the surface – as in the figure above.

This definition of curl is **independent of the choice of basis**. The usual Cartesian form for **curl** $\underline{A}$ can be recovered from this general definition by considering small rectangles in the $(\underline{e}_1 - \underline{e}_2)$, $(\underline{e}_2 - \underline{e}_3)$ and $(\underline{e}_3 - \underline{e}_1)$ planes respectively, but you are not required to prove this.

4.9.2 Cartesian form of Curl

Let P be a point with Cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangle $C = abcd$ of size $\delta_1 \times \delta_2$, area $\Delta S = \delta_1\,\delta_2$, in the $(\underline{e}_1 - \underline{e}_2)$ plane.



The line integral around C is given by the sum of four terms

$$\oint_C \underline{A}\cdot\underline{dr} \;=\; \int_a^b \underline{A}\cdot\underline{dr} + \int_b^c \underline{A}\cdot\underline{dr} + \int_c^d \underline{A}\cdot\underline{dr} + \int_d^a \underline{A}\cdot\underline{dr}$$

Since $\underline{r} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$, we have $\underline{dr} = \underline{e}_1\,dx$ along $d \to a$ and $c \to b$, and $\underline{dr} = \underline{e}_2\,dy$ along $a \to b$ and $d \to c$. Therefore

$$\oint_C \underline{A}\cdot\underline{dr} \;=\; \int_a^b A_2\,dy - \int_c^b A_1\,dx - \int_d^c A_2\,dy + \int_d^a A_1\,dx$$

For small δ_1 & δ_2, we can Taylor expand the integrands, viz

$$\begin{aligned}
\int_d^a A_1\,dx &= \int_d^a A_1(x, y_0 - \delta_2/2, z_0)\,dx \\
&= \int_{x_0-\delta_1/2}^{x_0+\delta_1/2} \left[A_1(x,y_0,z_0) - \frac{\delta_2}{2}\frac{\partial A_1(x,y_0,z_0)}{\partial y} + O(\delta_2^2) \right] dx \\
\int_c^b A_1\,dx &= \int_c^b A_1(x, y_0 + \delta_2/2, z_0)\,dx \\
&= \int_{x_0-\delta_1/2}^{x_0+\delta_1/2} \left[A_1(x,y_0,z_0) + \frac{\delta_2}{2}\frac{\partial A_1(x,y_0,z_0)}{\partial y} + O(\delta_2^2) \right] dx
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{\Delta S}\left[\int_d^a \underline{A}\cdot\underline{dr} + \int_b^c \underline{A}\cdot\underline{dr}\right] &= \frac{1}{\delta_1\,\delta_2}\left[\int_d^a A_1\,dx - \int_c^b A_1\,dx\right] \\
&= \frac{1}{\delta_1\delta_2}\int_{x_0-\delta_1/2}^{x_0+\delta_1/2}\left[-\delta_2\frac{\partial A_1(x,y_0,z_0)}{\partial y} + O(\delta_2^2)\right] dx \\
&\to -\frac{\partial A_1(x_0,y_0,z_0)}{\partial y} \quad \text{as} \quad \delta_1,\,\delta_2 \to 0
\end{aligned}$$

A similar analysis of the line integrals along $a \to b$ and $c \to d$ gives

$$\frac{1}{\Delta S}\left[\int_a^b \underline{A}\cdot\underline{dr} + \int_c^d \underline{A}\cdot\underline{dr}\right] \to \frac{\partial A_2(x_0,y_0,z_0)}{\partial x} \quad \text{as} \quad \delta_1,\,\delta_2 \to 0$$

Adding the results gives for our line integral definition of curl yields

$$\underline{e}_3\cdot\left(\underline{\nabla}\times\underline{A}\right) = \left(\underline{\nabla}\times\underline{A}\right)_3 = \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]\Bigg|_{(x_0,\,y_0,\,z_0)}$$

in agreement with our original definition in Cartesian coordinates.

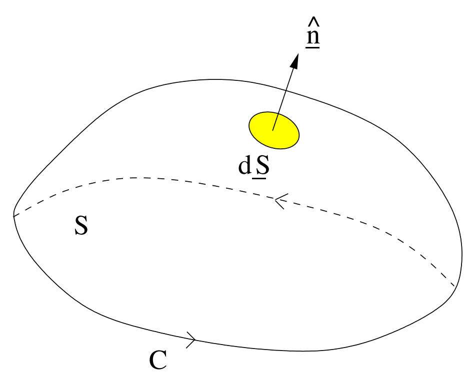
The other components of curl $\underline{A}$ can be obtained from similar rectangles in the $(\underline{e}_2 - \underline{e}_3)$ and $(\underline{e}_1 - \underline{e}_3)$ planes, respectively.

4.9.3 Stokes' Theorem

If S is an **open** surface, bounded by a simple **closed** curve C, and $\underline{A}$ is a vector field defined on S, then

$$\boxed{\oint_C \underline{A}\cdot\underline{dr} = \int_S \left(\underline{\nabla}\times\underline{A}\right)\cdot\underline{dS}}$$

where C is traversed in a right-hand sense about $\underline{dS}$. (As usual $\underline{dS} = \underline{\hat{n}}dS$ and $\underline{\hat{n}}$ is the unit normal to S).



Proof:

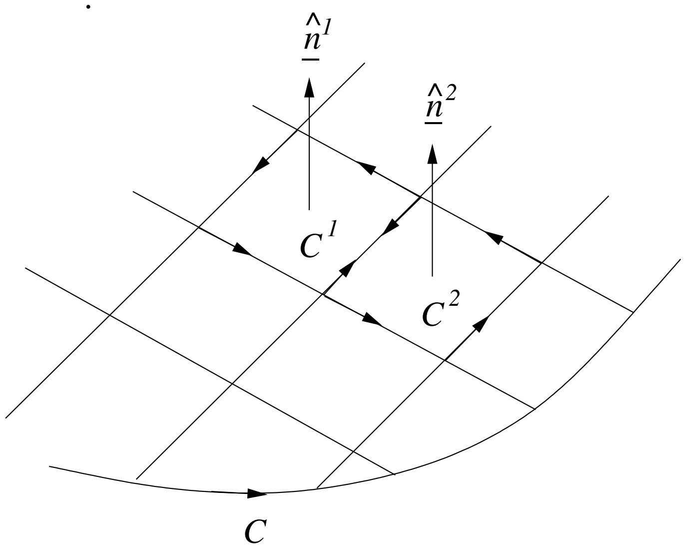
Divide the surface area S into N adjacent small surfaces as indicated in the diagram. Let $\underline{\Delta S^i} = \Delta S^i\, \underline{\hat{n}}^i$ be the vector element of area at $\underline{r}^i$. Using the integral definition of curl,

$$\underline{\hat{n}} \cdot \left(\text{curl } \underline{A}\right) \;=\; \underline{\hat{n}} \cdot \left(\underline{\nabla} \times \underline{A}\right) \;=\; \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \underline{A} \cdot \underline{dr}$$

we multiply by ΔS^i and sum over all i to get

$$\sum_{i=1}^{N} \left(\underline{\nabla} \times \underline{A}(\underline{r}^i)\right) \cdot \underline{\hat{n}}^i\, \Delta S^i \;=\; \sum_{i=1}^{N} \oint_{C^i} \underline{A} \cdot \underline{dr} + \sum_{i=1}^{N} \epsilon^i\, \Delta S^i$$

where C^i is the curve enclosing the area ΔS^i, and the quantity $\epsilon^i \to 0$ as $\Delta S^i \to 0$.



Since each small closed curve C^i is traversed in the same sense, then, from the diagram, all contributions to $\sum_{i=1}^{N} \oint_{C^i} \underline{A} \cdot \underline{dr}$ **cancel**, except on those curves where part of C^i lies on the curve C. For example, the line integrals along the common sections of the two small closed curves C^1 and C^2 **cancel exactly**. Therefore

$$\sum_{i=1}^{N} \oint_{C^i} \underline{A} \cdot \underline{dr} \;=\; \oint_C \underline{A} \cdot \underline{dr}$$

Hence

$$\oint_C \underline{A} \cdot \underline{dr} \;=\; \int_S \left(\underline{\nabla} \times \underline{A}\right) \cdot \underline{dS} \;=\; \int_S \underline{\hat{n}} \cdot \left(\underline{\nabla} \times \underline{A}\right)\, dS$$

Mathematical Note: For those worried about how to analyse 'the error term', note that for finite N, we can put an upper bound

$$\sum_{i=1}^{N} \epsilon^i \, \Delta S^i \;\leq\; S \max_i \; \{\epsilon^i\}$$

This tends to zero in the limit $N \to \infty$, because $\epsilon^i \to 0$ and S is finite.

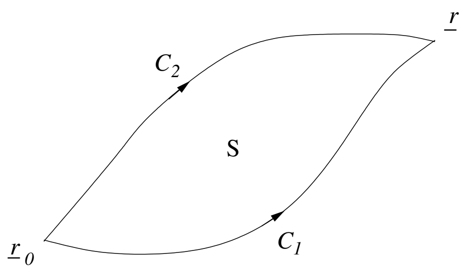
end of lecture 21

4.9.4 Applications of Stokes' Theorem

In Lecture 17 it was stated that if a vector field is irrotational (curl vanishes) then a line integral is independent of path. We can now prove this statement using Stokes' theorem.

Proof:

Let $\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0$ in R, and consider the **difference** of two line integrals from the point $\underline{r}_0$ to the point $\underline{r}$ along the two curves C_1 and C_2 as shown:



$$\int_{C_1} \underline{A}(\underline{r}') \cdot \underline{dr}' \;-\; \int_{C_2} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

We use $\underline{r}'$ as integration variable to distinguish it from the **limits** of integration $\underline{r}_0$ and $\underline{r}$.

We can rewrite this as the integral around the **closed** curve $C = C_1 - C_2$:

$$\int_{C_1} \underline{A}(\underline{r}') \cdot \underline{dr}' \;-\; \int_{C_2} \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}'$$

$$=\; \int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS} \;=\; 0$$

In the above, we have used Stokes' theorem to write the *line* integral of $\underline{A}$ around the closed curve $C = C_1 - C_2$, as the *surface* integral of $\underline{\nabla} \times \underline{A}$ over an open surface S bounded by C. This integral is zero because $\underline{\nabla} \times \underline{A} = 0$ everywhere in R. Hence

$$\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0 \quad \Rightarrow \quad \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; 0$$

for *any* closed curve C in R as claimed.

Clearly, the converse is also true i.e. if the line integral between two points is path independent then the line integral around any closed curve (connecting the two points) is zero. Therefore

$$0 = \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' = \int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS}$$

where we have used Stokes' theorem and since this holds for any S the field must be irrotational.

Planar Areas

Consider a planar surface in the $\underline{e}_1-\underline{e}_2$ plane and the vector field

$$\underline{A} = \frac{1}{2}\left[-y\underline{e}_1 + x\underline{e}_2\right] \ .$$

We find $\underline{\nabla} \times \underline{A} = \underline{e}_3$. Since a vector element of area normal to a planar surface in the $\underline{e}_1-\underline{e}_2$ plane is $\underline{dS} = dS\,\underline{e}_3$ we can obtain the area in the following way

$$\int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS} = \int_S \underline{e}_3 \cdot \underline{dS} = \int_S dS = S$$

Now we can use Stokes' theorem to find

$$\begin{aligned} S &= \oint_C \underline{A} \cdot \underline{dr} = \frac{1}{2}\oint_C (-y\underline{e}_1 + x\underline{e}_2) \cdot (\underline{e}_1 dx + \underline{e}_2 dy) \\ &= \frac{1}{2}\oint_C (x\,dy - y\,dx) \end{aligned}$$

where C is the closed curve bounding the surface.

e.g. To find the area inside the curve

$$x^{2/3} + y^{2/3} = 1$$

use the substitution $x = \cos^3\phi$, $y = \sin^3\phi$, $0 \le \phi \le 2\pi$ then

$$\frac{dx}{d\phi} = -3\cos^2\phi\,\sin\phi \quad ; \quad \frac{dy}{d\phi} = 3\sin^2\phi\,\cos\phi$$

and we obtain

$$\begin{aligned} S &= \frac{1}{2}\oint_C \left(x\frac{dy}{d\phi} - y\frac{dx}{d\phi}\right) d\phi \\ &= \frac{1}{2}\int_0^{2\pi} \left(3\cos^4\phi\sin^2\phi + 3\sin^4\phi\cos^2\phi\right) d\phi \\ &= \frac{3}{2}\int_0^{2\pi} \sin^2\phi\cos^2\phi\ d\phi = \frac{3}{8}\int_0^{2\pi} \sin^2 2\phi\,d\phi = \frac{3\pi}{8} \end{aligned}$$

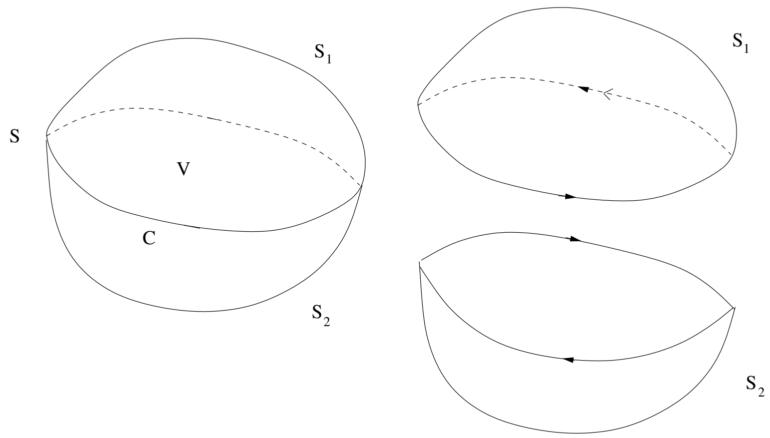
4.9.5 Example on joint use of Divergence and Stokes' Theorems

Example: show that $\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} \equiv 0$ independent of co-ordinate system:

Let S be a closed surface, enclosing a volume V. Applying the divergence theorem to $\underline{\nabla} \times \underline{A}$, we obtain

$$\int_V \underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{A}\right) \, dV \;=\; \int_S \left(\underline{\nabla} \times \underline{A}\right) \cdot \underline{dS}$$

Now divide S into two surfaces S_1 and S_2 with a **common** boundary C as shown below



Now use Stokes' theorem to write

$$\int_S \left(\underline{\nabla} \times \underline{A}\right) \cdot \underline{dS} \;=\; \int_{S_1} \left(\underline{\nabla} \times \underline{A}\right) \cdot \underline{dS} + \int_{S_2} \left(\underline{\nabla} \times \underline{A}\right) \cdot \underline{dS} \;=\; \oint_C \underline{A} \cdot \underline{dr} - \oint_C \underline{A} \cdot \underline{dr} \;=\; 0$$

where the second line integral appears with a minus sign because it is traversed in the **opposite** direction. (Recall that Stokes' theorem applies to curves traversed in the right hand sense with respect to the outward normal of the surface.)

Since this result holds for arbitrary volumes, we must have

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} \equiv 0$$

end of lecture 22